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ARTICLE PRINCIPAL

Recouvrant en partie

LA THESE DE DOCTORAT ES-SCIENCES MATHÉMATIQUES

présentée

A LA FACULTE DES SCIENCES

DE L'UNIVERSITE DE PARIS

par

John MADORE

pour obtenir le grade de

DOCTEUR ES-SCIENCES MATHÉMATIQUES

Sujet de la Thèse : GRAVITATIONAL RADIATION FROM A BOUNDED SOURCE.

Soutenue le 17 OCTO 1969

devant la Commission d'Examen

M. A. LICHNEROWICZ

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M. A. PAPAPETROU

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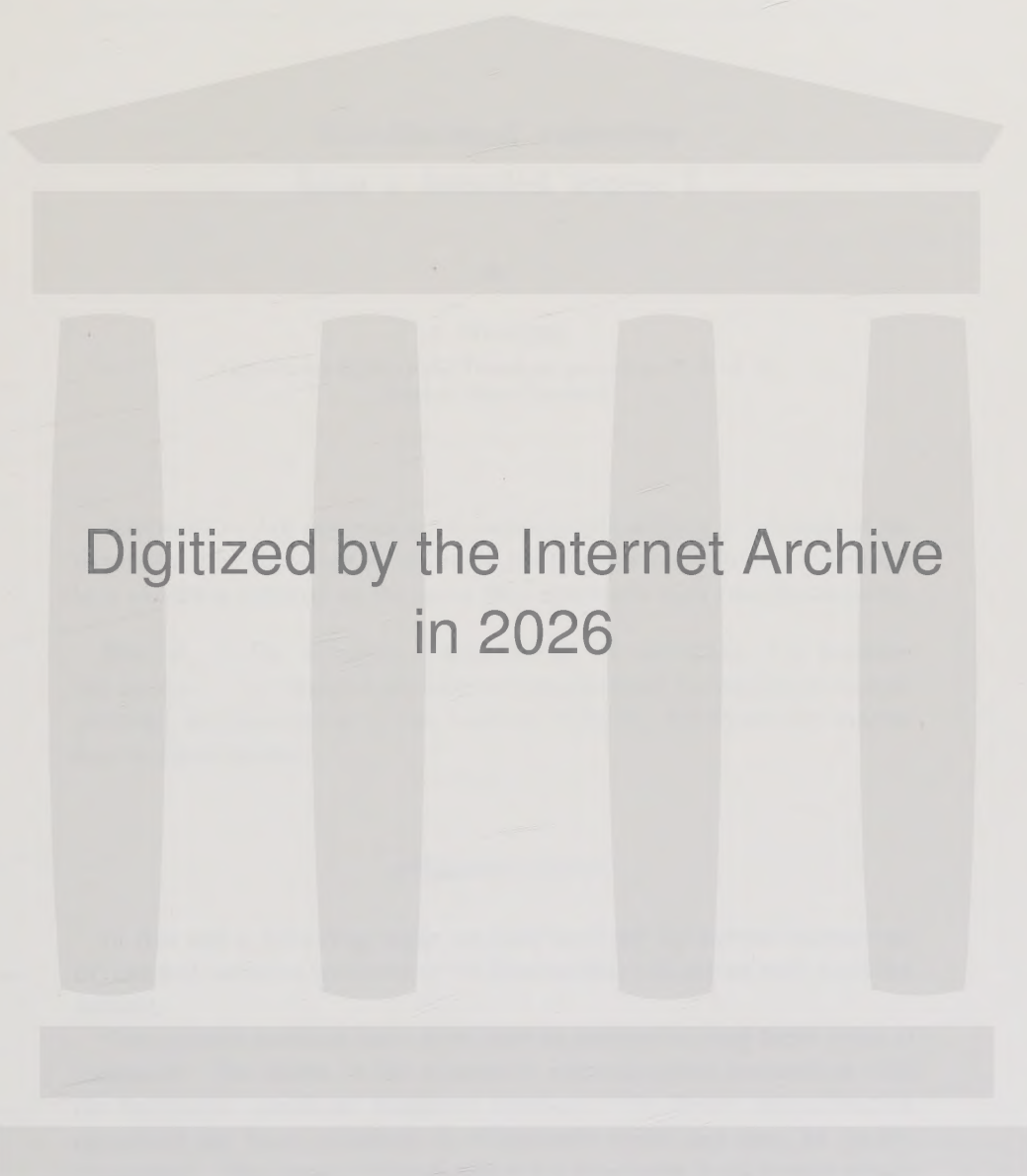


C'est un plaisir pour moi d'exprimer ma gratitude à Monsieur A. Papapetrou qui m'a proposé le sujet de ce travail et m'a fait part de nombreux conseils tout au long de son élaboration.

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TABLE DES MATIERES

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Gravitational radiation from a bounded source I

par

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ABSTRACT. — An obstruction is found to considering a solution of the linearized field equations in harmonic coordinates as a first approximation to a radiative solution of the exact field equations with bounded sources.

RÉSUMÉ. - On démontre l'impossibilité de considérer une solution des équations du champ linéarisées en coordonnées harmoniques comme première approximation à une solution radiative des équations exactes avec sources bornées.

INTRODUCTION

In this and a following paper we shall consider asymptotic expansions of retarded radiative solutions of the Einstein field equations with bounded sources.

Two general methods have been used to attempt to find these types of solutions. The oldest is the successive approximation procedure with the harmonic coordinate condition imposed. The linear approximation equations are wave equations in Minkowski space and may be readily integrated. The second method which has been used is the formal power series approach initiated by Bondi [1]. The metric is assumed to have an asymptotic expansion along a family of forward null hypersurfaces $u = \text{const.}$ in inverse powers of a radial parameter r and the field equations are formally integrated.

The partial solutions obtained by these two methods have a property in common which will be of interest to us here.

A retarded solution of the linearized field equations in harmonic coordinates possesses an asymptotic expansion in powers of $1/r$ along any regular family of forward Minkowski null cones. This expansion is the multipole expansion

The Bondi-type asymptotic solutions have a similar property. Consider in fact the coordinate system (x^i) obtained from a Bondi-type coordinate system (u, r, θ, ϕ) by the following coordinate transformation :

$$x^0 = u + r, \quad x^1 = r \sin \theta \cos \phi, \quad x^2 = r \sin \theta \sin \phi, \quad x^3 = r \cos \theta. \quad (1)$$

If we take a solution of the type obtained by Bondi and Sachs [1], [2] and express it in the coordinate system (x^i) we find that the components of the metric tensor approach the standard Minkowski metric asymptotically as r tends to infinity. We find also that the hypersurfaces $u = \text{const.}$ along which the components of the metric tensor are given as an asymptotic expansion in powers of $1/r$ form a family of forward Minkowski null cones of the coordinate system (x^i) .

The same remarks apply to the asymptotic solutions found by Newman and Penrose [3] in a slightly different type of coordinate system. The coordinate systems obtained by (1) from Newman-Penrose-type coordinate systems have been used by Papapetrou [4] to study the asymptotic properties of exact solutions of the field equations.

We see then that all of these coordinate systems have the property that the components of the metric tensor possess an asymptotic expansion along a family of hypersurfaces which are either the Minkowski null cones (Papapetrou type) or which are the Minkowski null cones of a simply related coordinate system through the coordinate transformation (1) (Newman-Penrose type and Bondi-Sachs type).

What we shall do in Part I is consider all coordinate systems in which the components of a given radiative solution of the field equations with bounded sources possess an asymptotic expansion in inverse powers of a radial parameter r along a family of forward Minkowski null hypersurfaces. We shall find a necessary and sufficient condition which the components of the metric tensor must satisfy in order that such a expansion exist.

When we apply these considerations to the retarded solution of the linearized field equations in harmonic coordinates we find that the condition is not satisfied. We have then a condition which must be satisfied by a solution to the exact equations but which is not by the linear approxi-

mation solution. Using this fact we shall show that it is impossible to consider a retarded solution of the linearized field equations in harmonic coordinates as a first approximation to a retarded radiative solution of the exact field equations with bounded sources.

There are four sections. In Section I a decomposition is given of an arbitrary symmetric matrix with respect to two Minkowski null vectors.

In sections II and III we use the field equations

$$R_{ij} = 0,$$

which are valid outside of the world-tube of the sources. In Section II we do not assume an expansion in powers of $1/r$ along any set of hypersurfaces. We only assume that the field becomes Minkowskian as $1/r$ in the limit as r tends to infinity.

The main section is III. We here assume an asymptotic expansion in powers of $1/r$ along a family of forward Minkowski null cones and consider the conclusions which can be drawn from setting $R_{ij}^{(2)}$, the coefficient of $1/r^2$ in the asymptotic expansion of the Ricci tensor, equal to zero.

In the last section we consider the method of successive approximations using the harmonic coordinate condition.

I

Let V_4 be a Space-Time whose sources are restricted to some finite region of space. We shall assume that V_4 admits global coordinate systems

$$V_4 \xrightarrow[\sim]{\phi} \mathbb{R}^4$$

and we shall restrict our attention to the subset Φ'' of these ϕ such that the components of the metric tensor are of the form

$$g_{ij} = \eta_{ij} + \frac{h_{ij}(u, r, \theta, \phi)}{r} \quad (2)$$

where (r, θ, ϕ) is the polar coordinate system for $\mathbb{R}^3$ and $u = x^0 - r$ (We shall later define the sets of coordinate systems Φ' and Φ). Latin indices take the values $(0, 1, 2, 3)$ and η_{ij} is the standard Minkowski metric of signature -2 . In the following we shall use η_{ij} to raise and lower all indices. The coordinate functions (x^i) are related to the functions (u, r, θ, ϕ) by the transformation (1) of the Introduction. We use the latter only

as expansion parameters; (u, r, θ, ϕ) considered as a coordinate system does not belong to the set Φ'' . The hypersurfaces $u = \text{const.}$ are the family of forward Minkowski null cones whose vertices lie on the world-line of the origin.

The functions h_{ij} as well as their first and second derivatives are assumed to be bounded functions of r in the limit as r tends to infinity. Since the sources are bounded we may impose also that the time axes be such that the Ricci tensor vanishes outside of a timelike cylinder $r = \text{const.}$ Our group G'' of admissible coordinate transformations will be the set of maps $\mathbb{R}^4 \rightarrow \mathbb{R}^4$ such that $G'' \circ \Phi'' \subset \Phi''$. Notice that G'' contains only those Lorentz transformations which do not change the direction of the time axes.

We wish to consider retarded radiative solutions of the fields equations (We shall define later what we mean by a retarded solution). We shall investigate the conditions which the coefficient of $1/r$ on the right-hand side of equation (2) must satisfy in order that the left-hand side be a radiative solution of the field equations.

The principal tool of our investigations is a decomposition of an arbitrary symmetric matrix with respect to the pair of associated Minkowski null vectors ξ^i and ζ^i , given by

$$\begin{aligned}\xi^i &= (1, x^a/r), \\ \zeta^i &= (-1, x^a/r).\end{aligned}$$

Greek indices take the values (1, 2, 3).

Define a symmetric projection operator π_{ij} by

$$\pi_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & \eta_{\alpha\beta} + \xi_\alpha \zeta_\beta \end{bmatrix}.$$

π_{ij} satisfies the following relations:

$$\eta^{ij}\pi_{ij} = 2, \quad \pi_{ij}\xi^j = 0, \quad \pi_{ij}\zeta^i = 0, \quad \pi_{ij}\pi_{jk} = \pi_{ik}. \quad (3)$$

Let h_{ij} be an arbitrary symmetric matrix. Set

$$h = \eta^{ij}h_{ij}, \quad h(\xi, \zeta) = h_{ij}\xi^i\zeta^j.$$

Define the quantities $\hat{h}$, K , H , $\hat{K}$, $\hat{h}_i$, H_i by the equations

$$\hat{h} = \frac{1}{4}h(\zeta, \zeta), \quad K = \frac{1}{4}h(\xi, \xi), \quad H = \frac{1}{4}h(\xi, \zeta), \quad \hat{K} = h + 4K, \quad (4a)$$

$$\hat{h}_i = \frac{-1}{2} h_{ij} \zeta^j - \left\{ \hat{h} \zeta_i - K \zeta_i \right\} \quad (4b)$$

$$H_i = \frac{-1}{2} h_{ij} \xi^j - K \xi_i - H \zeta_i. \quad (4c)$$

The quantities $\hat{h}_i$ and H_i satisfy the relations

$$\hat{h}_i \xi^i = \hat{h}_i \zeta^i = H_i \xi^i = H_i \zeta^i = 0. \quad (5a)$$

We have also

$$\hat{h}_0 = H_0 = 0. \quad (5b)$$

Therefore we find

$$\pi_{ij} \hat{h}^j = \hat{h}_i, \quad \pi_{ij} H^j = H_i. \quad (5c)$$

Define $\hat{h}_{ij}$ by the equation

$$h_{ij} = h_{ij} + 2\hat{h}_{(i} \zeta_{j)} + 2H_{(i} \zeta_{j)} + \frac{K}{2} \pi_{ij} + \hat{h} \zeta_i \zeta_j + 2K \xi_{(i} \zeta_{j)} + H \zeta_i \zeta_j. \quad (6)$$

Multiplying both sides of this equation respectively by ξ^i , ζ^i , η^{ij} , we find that $\hat{h}_{ij}$ satisfies the relations

$$\hat{h}_{ij} \xi^j = h_{ij} \zeta^j = \hat{h}_i = 0. \quad (7a)$$

We find also that

$$\hat{h}_{0i} = 0, \quad \pi_{ij} \hat{h}^j = \hat{h}_i. \quad (7b)$$

Equation (6) is the decomposition of h_{ij} with respect to ξ^i and ζ^i . We have four functions $\hat{h}$, K , H , $\hat{K}$, two Cartesian vectors $\hat{h}_i$, H_i each with two independent components and a symmetric Cartesian tensor $\hat{h}_{ij}$ with two independent components, giving a total of ten components for h_{ij} .

The decomposition (6) is similar to the Fourier transform of the spin decomposition given by Arnowitt, Deser and Misner [5]. We are here working in the position variable space not in the momentum space. However, we are primarily interested in the leading term of a spherical wave far from the sources and in this wave zone a spin decomposition is equivalent to the algebraic decomposition we have given. For example if f^i is a vector field of the form

$$f^i = \frac{f_1^i(u, \theta, \phi)}{r} + O(r^{-2}),$$

then we have

$$f_{,i}^i = \frac{\partial f_1^i}{\partial u} \frac{\xi_i}{r} + O(r^{-2}).$$

We see that to within terms of order $O(r^{-2})$ the differential condition

$$f_{,i}^i = 0$$

is equivalent to the algebraic condition

$$\frac{\partial f_1^i}{\partial u} \xi_i = 0.$$

A similar decomposition may be given with respect to an arbitrary pair of associated vectors

$$p^i = (p^0, p^a), \quad q^i = (-p^0, p^a),$$

which are time-like or null. Formula (6) remains valid with (ξ^i, ζ^i) replaced by (p^i, q^i) but the expressions on the right-hand side of the defining relations (4) are more complicated.

II

In an admissible coordinate system $\phi \in \Phi''$ the components of the metric tensor are of the form (I-2).

Set

$$\dot{h}_{ij} = \frac{\partial h_{ij}}{\partial u}, \quad h'_{ij} = \frac{\partial h_{ij}}{\partial r}.$$

Then, since

$$\frac{\partial r}{\partial x^a} = -\xi_a, \quad \frac{\partial u}{\partial x^i} = \xi_i,$$

we have

$$g_{ij,kl} = \frac{\dot{h}_{ij}}{r} \xi_k \xi_l - \frac{2\dot{h}'_{ij}}{r} \xi_{(k} \delta_{l)}^a \xi_a + \frac{h''_{ij}}{r} \delta_k^a \delta_l^b \xi_a \xi_b + O(r^{-2}), \quad (1a)$$

where we define the symmetric product of two vectors a_i and b_i as

$$a_i b_j = \frac{1}{2} (a_i b_j + a_j b_i).$$

Using the fact that

$$\xi_i + \zeta_i = 2\delta_i^a \xi_a$$

(1a) may be written as

$$g_{ij,kl} = \frac{1}{4r} [(4\dot{h}_{ij} - 4\dot{h}'_{ij} + \dot{h}''_{ij}) \xi_k \xi_l - 2(2\dot{h}'_{ij} - h''_{ij}) \xi_{(k} \zeta_{l)} + h''_{ij} \zeta_k \zeta_l] + O(r^{-2}). \quad (1b)$$

Equation (I-2) defines a symmetric matrix of functions h_{ij} which we may decompose according to equation (I-6). Since the Ricci tensor vanishes for large values of r , the elements of this decomposition will not be arbitrary but will be forced to satisfy certain equations. To obtain those equations which will interest us here in Section II it is sufficient to consider the leading term of the Ricci tensor. That is, it is sufficient to impose

$$R_{ij} = O(r^{-2}). \quad (2)$$

From equation (1b) we find the following expression for the Riemann tensor :

$$\begin{aligned} R_{ijkl} = & \frac{1}{2r} (4\xi_{[i}\ddot{h}_{j]lk}\xi_{\eta]} - 4\xi_{[i}\dot{h}'_{j]lk}\xi_{\eta]} + \xi_{[i}h''_{j]lk}\xi_{\eta]} \\ & - \frac{1}{2r} (2\xi_{[i}\dot{h}'_{j]lk}\xi_{\eta]} + 2\xi_{[i}\dot{h}'_{j]lk}\xi_{\eta]} - \xi_{[i}h''_{j]lk}\xi_{\eta]} - \xi_{[i}h''_{j]lk}\xi_{\eta]} \\ & + \frac{1}{2r} \xi_{[i}h''_{j]lk}\xi_{\eta]} + O(r^{-2}). \end{aligned} \quad (3)$$

Therefore the Ricci tensor is given by

$$\begin{aligned} R_{jk} = & \dot{h}\xi_j\xi_k - 2\dot{h}_{(j}\xi_{k)} \\ & - 2\dot{h}'_{\alpha}\xi_{\alpha}\delta_{(j}^{\alpha}\xi_{k)} + 2\dot{h}'_{(j}\xi_{k)}\xi_{\alpha} + 2\xi^{\alpha}\dot{h}'_{\alpha(j}\xi_{k)} + 2\dot{h}'_{jk} \\ & + h''\delta_j^{\alpha}\delta_k^{\beta}\xi_{\alpha}\xi_{\beta} - 2\xi^{\beta}h''_{\beta(j}\delta_{k)}^{\alpha}\xi_{\alpha} - h''_{jk} + O(r^{-1}), \end{aligned} \quad (4)$$

where for simplicity we have set $h_{ij}\xi^j = h_i$. If we decompose h_{ij} according to equation (I-6), equation (4) becomes

$$\begin{aligned} 2rR_{jk} = & 2\ddot{h}_{jk} - \dot{h}''_{jk} \\ & + 4\dot{H}_{(j}\xi_{k)} - 4\dot{H}'_{(j}\xi_{k)} + H''_{(j}\xi_{k)} + 2\dot{h}'_{(j}\xi_{k)} - \dot{h}''_{(j}\xi_{k)} \\ & + 2\dot{H}'_{(j}\xi_{k)} - H''_{(j}\xi_{k)} + \dot{h}''_{(j}\xi_{k)} \\ & + (\dot{K}' - \dot{K}''/2)\pi_{ij} \\ & + (\ddot{K} - \dot{K}' + \dot{K}''/4)\xi_j\xi_k \\ & + \left(4\dot{H} - \dot{h}' - 4\dot{H}' + \dot{h}'' + H'' + \frac{h''}{2}\right)\xi_{(j}\xi_{k)} \\ & + \frac{\dot{K}''}{4}\xi_j\xi_k + O(r^{-1}). \end{aligned} \quad (5)$$

Therefore after a short calculation we see that equation (2) is equivalent to the following set of equations :

$$2\dot{h}_{ij} - \dot{h}_{ij}'' = O(r^{-1}), \quad (6a)$$

$$4\dot{H}_j - 4\dot{H}_j' + H_j'' + 2\dot{h}_j' - \dot{h}_j'' = O(r^{-1}), \quad (6b)$$

$$2\dot{H}_j' - H_j'' + h_j'' = O(r^{-1}), \quad (6c)$$

$$\ddot{K} = \dot{K}' = \dot{K}'' = O(r^{-1}), \quad 4\dot{H} - \dot{h}' - 4\dot{H}' + \dot{h}'' + H'' + \frac{h''}{2} = O(r^{-1}). \quad (6d)$$

Using these equations, (3) reduces to the form

$$R_{ijkl} = \frac{1}{2r} (4\xi_{[i}\ddot{h}_{j]lk}\xi_{l]} - 2\xi_{[i}\dot{h}_{j]lk}'\xi_{l]} + \xi_{[i}\ddot{h}_{j]lk}''\xi_{l]}) + O(r^{-2}). \quad (7)$$

This yields us the following result: the leading term of the components of the Riemann tensor in an admissible coordinate system depends only on the part $\dot{h}_{ij}$ of the matrix h_{ij} . This is true without imposing an outgoing (or incoming) radiation condition. We see from the form of (7) that the leading term of the Riemann tensor vanishes if the matrix $\dot{h}_{ij}$ does not depend on u or r .

Let us consider now in more detail equation (6a). This equation has two particular classes of solutions which yield simplified expressions for the leading term of the components of the Riemann tensor. The first class consists of those matrices $\dot{h}_{ij}^{(r)}$ which satisfy

$$\dot{h}_{ij}^{(r)} = \frac{f_{ij}(u, r, \theta, \phi)}{r}, \quad (8)$$

where f_{ij} is an arbitrary matrix of functions which are bounded along with their first derivatives with respect to r as r tends to infinity. A particular subset of these solutions is given by

$$\dot{h}_{ij}^{(r)} = \dot{h}_{ij}^{(1)}(u, \theta, \phi) + \frac{\dot{h}_{ij}^{(2)}(u, \theta, \phi)}{r} + O(r^{-2}), \quad (9)$$

where $\dot{h}_{ij}^{(1)}$ and $\dot{h}_{ij}^{(2)}$ are arbitrary functions of (u, θ, ϕ) . These matrices $\dot{h}_{ij}^{(1)}$ and $\dot{h}_{ij}^{(2)}$ as well as the matrix f_{ij} given above in equation (8) must of course satisfy the algebraic conditions (I-7) at each point.

If $\dot{h}_{ij}^{(r)}$ is a solution of (6a) of the form (8) then the corresponding expression for the leading term of the components of the Riemann tensor given by equation (7) becomes

$$R_{ijkl} = \frac{2}{r} \xi_{[i}\ddot{h}_{j]lk}^{(r)}\xi_{l]} + O(r^{-2}). \quad (10)$$

The leading term is of type N with ξ_i the principal null vector (ξ_i are the components in the particular coordinate system considered of a vector whose norm tends to zero as $1/r$ in the limit as r tends to infinity).

Condition (8) may be considered as a weakened form of the Sommerfeld radiation condition.

Another particular class of solutions to equation (6a) consists of those matrices $\hat{h}_{ij}^{(a)}$ which satisfy

$$2\dot{\hat{h}}_{ij}^{(a)} - \hat{h}_{ij}^{\prime(a)} = \frac{f_{ij}(u, r, \theta, \phi)}{r}, \quad (11)$$

where f_{ij} is an arbitrary matrix of functions of the same type as in (8). A particular subset of these solutions is given by

$$\hat{h}_{ij}^{(a)} = \hat{h}_{ij}^{(1)}(u + 2r, \theta, \phi) + \frac{\hat{h}_{ij}^{(2)}(u + 2r, \theta, \phi)}{r} + O(r^{-2}), \quad (12)$$

where $\hat{h}_{ij}^{(1)}$ and $\hat{h}_{ij}^{(2)}$ are of the same type as in (9).

If $\hat{h}_{ij}^{(a)}$ is a solution of (6a) of the form (11) then the corresponding expression for the leading term of the components of the Riemann tensor given by equation (7) becomes

$$R_{ijkl} = \frac{2}{r} \zeta_{[i} \dot{\hat{h}}_{jk}^{\prime(a)} \zeta_{l]} + O(r^{-2}). \quad (13)$$

The leading term is again of type N but with principal null vector ζ_i .

Equation (7) yields the result that the $1/r$ term of the Riemann tensor is always of type N or I (if it does not vanish). No combination of outgoing or incoming radiation can yield an intermediate type. In fact, if we define A_{ij} and B_{ij} by

$$A_{ij} = 2(2\dot{\hat{h}}_{ij} - \hat{h}_{ij}'), \quad B_{ij} = \hat{h}_{ij}'', \quad (14)$$

then equation (7) may be written as follows:

$$R_{ijkl} = \frac{1}{2r} (\zeta_{[i} A_{j]kl} \zeta_{l]} + \zeta_{[i} B_{j]kl} \zeta_{l]}) + O(r^{-2}). \quad (15)$$

If either A_{ij} or B_{ij} vanishes as $r^{-\alpha}$ the leading term is obviously of type N. Suppose now that, for example, $\lim_{r \rightarrow \infty} A_{ij}$ does not vanish and that the leading term is algebraically special. We wish to show that this implies that B_{ij} must vanish as $r^{-\alpha}$ ($\alpha > 0$).

If the leading term is algebraically special then there exists a non-zero null vector field a_i which satisfies the following equation:

$$a_{[m} R_{j]ikl} a^l a^i = O(r^{-(1+\alpha)}). \quad (16)$$

We may decompose a_i as follows:

$$a_i = -\frac{1}{2}(a \cdot \xi)\xi_i - \frac{1}{2}(a \cdot \zeta)\zeta_i + \hat{a}_i \quad (17)$$

where $\hat{a}_i$ satisfies the conditions (I-5). Using this and the expression (15) for the Riemann tensor we find that (16) becomes

$$\begin{aligned} a_{[m} R_{j]ikl} a^l a^i = & -\frac{1}{8r} \{ a_{[m} A_{j]ik} (a \cdot \xi)^2 + a_{[m} B_{j]ik} (a \cdot \zeta)^2 \\ & - \hat{a}^i A_{ik} a_{[m} \xi_{j]} (a \cdot \xi) - \hat{a}^i A_{i[j} a_{m]} \xi_k (a \cdot \xi) \\ & - \hat{a}^i B_{ik} a_{[m} \zeta_{j]} (a \cdot \zeta) - \hat{a}^i B_{i[j} a_{m]} \zeta_k (a \cdot \zeta) \\ & + A_{il} \hat{a}^l \hat{a}^i \xi_{[j} a_{m]} \zeta_k + B_{il} \hat{a}^l \hat{a}^i \zeta_{[j} a_{m]} \xi_k \} + O(r^{-(1+\alpha)}). \end{aligned} \quad (18)$$

If (16) is to be satisfied then the leading term of (18) must vanish as $r^{-\alpha}$. We may decompose this term with respect to the index k as we did a_i above in (17). This yields three equations

$$(a \cdot \xi)^2 + a_{[m} B_{j]ik} (a \cdot \zeta)^2 - a_{[m} \xi_{j]} (a \cdot \xi) \hat{a}^i A_{ik} - a_{[m} \zeta_{j]} (a \cdot \zeta) \hat{a}^i B_{ik} = O(r^{-\alpha}), \quad (19a)$$

$$A_{il} \hat{a}^l \hat{a}^i \xi_{[j} a_{m]} - \hat{a}^i A_{i[j} a_{m]} \xi_k = O(r^{-\alpha}), \quad (19b)$$

$$B_{il} \hat{a}^l \hat{a}^i \zeta_{[j} a_{m]} - \hat{a}^i B_{i[j} a_{m]} \zeta_k = O(r^{-\alpha}). \quad (19c)$$

If we denote asymptotic parallelism by the symbol $\sim$, we have from (19) the following conditions on a_i :

$$a_j \sim A_{jk} \lambda^k (a \cdot \xi)^2 + B_{jk} \lambda^k (a \cdot \zeta)^2 - \xi_j (a \cdot \xi) \hat{a}^i A_{ik} - \zeta_j (a \cdot \zeta) \hat{a}^i B_{ik} \lambda^k, \quad (20a)$$

$$a_j \sim A_{il} \hat{a}^l \hat{a}^i \xi_j - \hat{a}^i A_{ij} (a \cdot \xi), \quad (20b)$$

$$a_j \sim B_{il} \hat{a}^l \hat{a}^i \zeta_j - \hat{a}^i B_{ij} (a \cdot \zeta), \quad (20c)$$

where λ^i is an arbitrary vector field. We have from (17) the equation

$$\hat{a}^2 - (a \cdot \xi)(a \cdot \zeta) = O(r^{-1}) \quad (21)$$

The following diagram summarizes the various possibilities. The numbers beside the arrows indicate the formulae which are involved in

the implication. Note that from their definition (14), A_{ij} and B_{ij} satisfy the algebraic conditions (I. 7).

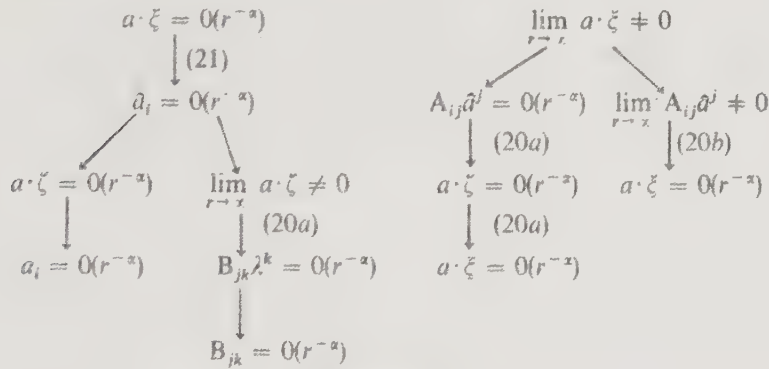


DIAGRAM 1.

The only consistent possibility is

$$B_{ij} = O(r^{-2}).$$

We see then that if the leading term is algebraically special it must be of type N.

III

We now assume that for a given bounded source configuration there is always a retarded solution of the field equations and from now on we shall restrict our considerations to these solutions. We define a retarded solution as a solution such that there exists a coordinate system $\phi \in \Phi''$ with the components g_{ij} of the metric tensor of the form

$$g_{ij} = \eta_{ij} + \frac{h_{ij}^{(1)}(u, \theta, \phi)}{r} + \frac{h_{ij}^{(2)}(u, \theta, \phi)}{r^2} + O(r^{-3}). \quad (1a)$$

That is, such that

$$h_{ij}(u, r, \theta, \phi) = h_{ij}^{(1)}(u, \theta, \phi) + \frac{h_{ij}^{(2)}(u, \theta, \phi)}{r} + O(r^{-2}). \quad (1b)$$

Since we shall no longer have occasion to use $h_{ij}(u, r, \theta, \phi)$, to alleviate the formulae we shall in the following drop the superscript on $h_{ij}^{(1)}(u, \theta, \phi)$. We define the coordinate set Φ' as the subset of Φ'' consisting of all ϕ such that (1) is satisfied.

Since all the information concerning radiation is contained in the

leading term of the asymptotic expansion, one could reasonably hope to be able to decide whether one had a retarded solution or not simply by regarding this term. We know that if Φ' is non-void then the leading $(1/r)$ term of the Riemann tensor is of type N [2]. It would be nice to be able to prove a converse to this result, that is, to prove that if the leading term of the Riemann tensor is of type N with a forward principal null vector then Φ' is non-void.

For the rest of this section we fix a particular coordinate system $\phi \in \Phi'$. Write (1a) in the form

$$g_{ij} = \eta_{ij} + \frac{h_{ij}(u, \theta, \phi)}{r} + \frac{\tilde{h}_{ij}(u, r, \theta, \phi)}{r^2}. \quad (2)$$

It is important that the following inequality be satisfied:

$$|\tilde{h}_{ij}(u, r, \theta, \phi)| \leq K |h_{ij}(u, \theta, \phi)| \quad (3)$$

where K is an arbitrary constant. It is this inequality that assures that the field variables tend uniformly to their Minkowski limit for all values of u as r tends to infinity. We shall not use (3) explicitly in what follows so we have not included it in the definition of Φ' . Strictly speaking however one cannot call a solution retarded unless (3) is satisfied for at least one coordinate system in Φ' . There is an interesting counter-example in cosmodynamics furnished by the everywhere regular vacuum solutions considered for example by Synge [6]. These solutions are symmetric under time inversion and so cannot represent everywhere outgoing radiation. They possess however for all values of (u, θ, ϕ) an expansion of the form (1) in powers of $1/r$ (*). An inspection of the coefficients of $1/r$ and $1/r^2$ shows that for arbitrarily large but fixed r the second term becomes dominant over the first for u tending to $-\infty$ and condition (3) is not satisfied for any finite K .

We now proceed as in the previous section and decompose the matrices of functions h_{ij} and $h_{ij}^{(2)}$ with respect to ξ^i and ζ^i . We do not claim that the elements of the decomposition of $h_{ij}^{(2)}$, which we distinguish with a superscript, have any physical significance; we use them only for convenience of calculation.

Since we have an asymptotic expansion for the metric tensor we shall also have one for the Ricci tensor:

$$R_{ij} = \frac{R_{ij}^{(1)}}{r} + \frac{R_{ij}^{(2)}}{r^2} + O(r^{-3}). \quad (4)$$

(*) I wish to thank M. Papapetrou for bringing this to my attention.

Setting $R_{ij}^{(1)}$ equal to zero proceeds as in the previous section and we obtain the set of equations (II-6) simplified by the fact now $h'_{ij} = 0 - h_{ij}$ does not depend on r . Equations (II-6) become

$$\dot{H}_j = \dot{H} = \ddot{K} = 0, \quad (5)$$

and since we are here considering a particular solution of (II-8) the leading term of the Riemann tensor is given by equation (II-10). With equations (5) satisfied, the Ricci tensor is therefore of the form

$$R_{ij} = \frac{R_{ij}^{(2)}}{r^2} + O(r^{-3}). \quad (6)$$

We shall now impose the condition

$$R_{ij}^{(2)} = 0, \quad (7)$$

and consider what restrictions this places on the elements of the decomposition of h_{ij} and $h_{ij}^{(2)}$. A short calculation gives that the system (7) is equivalent to the following set of equations (The details are given in the Appendix):

$$\begin{aligned} \dot{H}^{(2)} &= \dot{K}, \quad \dot{H} = 0, \quad \dot{K} = 0, \quad \dot{H}_i = 0, \\ \ddot{K}^{(2)} - 2\dot{h} - 2\dot{h}_{|a}^a - \ddot{h}_{ij}\dot{h}^{ij} - \frac{\dot{h}_{ij}}{2}\dot{h}^{ij} - 4\dot{K} &= 0, \\ h_{j|a}^a + 2\dot{h}_j - 2\dot{H}_j^{(2)} + 2\ddot{h}_{jk}H^k + 2\dot{K}_{|a}\delta_j^a &= 0, \end{aligned} \quad (8)$$

and

$$H\ddot{h}_{ij} = 0. \quad (9)$$

The most important of these equations is equation (9). It states that for the field equations to be satisfied either $\ddot{h}_{ij} = 0$ or $H = 0$.

From equation (II-10) we see that $\ddot{h}_{ij} = 0$ if and only if the $1/r$ term of the Riemann tensor vanishes. We now consider under what conditions the function H vanishes. Let $u = \text{const.}$ be a fixed Minkowski coordinate null cone and let $u + \omega = \text{const.}$ be a (with respect to the Riemannian metric) null hypersurface. Because of the expansion (1) for the components of the metric tensor, the Minkowski cone will be asymptotically tangent to a null hypersurface if and only if we can choose ω to be asymptotically of the form

$$\omega = \frac{\omega_1}{r} + O(r^{-2}) \quad (10)$$

where ω_1 is a function of (u, θ, ϕ) . Suppose this to be the case and let

p_i be the normal to the hypersurface $u + \omega = \text{const.}$ Then p_i is of the form

$$p_i = \xi_i + \frac{\dot{\omega}_1}{r} \xi_i + O(r^{-2}).$$

Since p_i is a null vector we have

$$\left(\eta^{ij} - \frac{h^{ij}}{r}\right) \left(\xi_i + \frac{\dot{\omega}_1}{r} \xi_i\right) \left(\xi_j - \frac{\dot{\omega}_1}{r} \xi_j\right) = O(r^{-2}), \quad (11)$$

and therefore $H = 0$.

Conversely if $H = 0$ an ω can be found of the form (10) by assuming an expansion in powers of $1/r$ and explicitly calculating the coefficients. If H does not vanish we may consider the hypersurfaces given by

$$\omega = 2H \log r + O(r^{-1}).$$

From the field equation (8b) we saw that H is independent of u . We shall see in Part II that H is a constant. Therefore one can show (as above for the case where $H=0$) that there exists a null hypersurface asymptotically of the form $u + 2H \log r + O(r^{-1}) = \text{const.}$

We have therefore shown that a necessary and sufficient condition for the Minkowski cones $u = \text{const.}$ to be asymptotically tangent to a null hypersurface is that H vanish. If H does not vanish then the cones diverge from all null hypersurfaces as $2H \log r$ in the limit as r tends to infinity.

Referring back to equation (9) we see that we have proven the following: the $1/r$ term in the asymptotic expansion of the Riemann tensor is of type N or vanishes according to whether the Minkowski cones $u = \text{const.}$ are asymptotically tangent to null hypersurfaces or not. Therefore if the

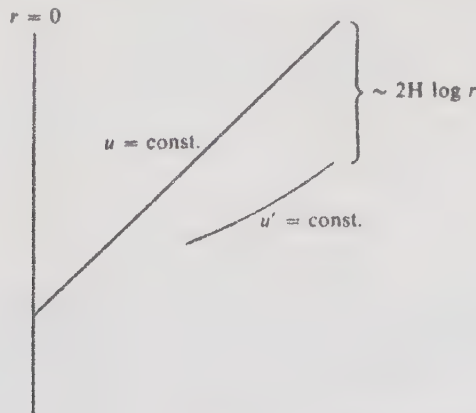


FIG. 1. — $u' = \text{const.}$ is a null hypersurface.

components of a metric tensor which is a radiative solution of the field equations possess an asymptotic expansion in powers of $1/r$ along the cones $u = \text{const.}$, then these cones must be asymptotically tangent to null hypersurfaces. Define the subset Φ of Φ' as the set of ϕ such that H vanishes. One may formulate the consequence of (9) as follows: if the $1/r$ term of the Riemann tensor does not vanish then

$$\Phi = \Phi'.$$

We have considered so far asymptotic expansions of the components of the metric tensor only along the particular family of forward Minkowski null cones $u = \text{const.}$ This restriction may be weakened. Suppose we have an asymptotic expansion of the components of the metric tensor in a coordinate system (x^i) along a regular family of hypersurfaces $u' = \text{const.}$ which has the following property: we can choose functions (r', θ', ϕ') such that the coordinate system (x'^i) constructed by formula (1) of the Introduction is in Φ . For example any regular family of Minkowski null hypersurfaces of the coordinate system (x^i) is a family which has this property. Let G be the group of coordinate transformations which takes Φ into itself: $G \circ \Phi \subset \Phi$. The two coordinate systems (x^i) and (x'^i) are connected by an element of G . It is easy to see (and we shall show in Part II) that H is invariant under the group G . Since $u' = \text{const.}$ is the family of forward Minkowski null cones of the coordinate system (x'^i) whose vertices lie on the world-line of the origin, we may conclude that

$$H = H' = 0.$$

We see therefore that the vanishing of H is a necessary and sufficient condition which the components of a metric tensor which is a radiative solution of the field equations with bounded sources, must satisfy in order that they possess an asymptotic expansion in inverse powers of r along a family of Minkowski null hypersurfaces.

IV

In conclusion we shall discuss the harmonic coordinate system in the first approximation. We shall discuss these coordinate systems in general in Part II [7]. Let g_{ij} be the components of the metric in a given harmonic coordinate system. We have then

$$g_{ij} = \eta_{ij} + k_{ij} \quad (1)$$

where $|k_{ij}| < \epsilon$. Define

$$\psi_{ij} = k_{ij} - \eta^{\mu\nu} k_{\mu\nu} \eta_{ij} / 2 \quad (2)$$

and let T_{ij} be the components of the matter tensor. If we neglect quadratic terms then the field equations

$$G_{ij} = -8\pi\kappa T_{ij}$$

may be explicitly integrated to give the retarded solution

$$\psi_{ij}(x) = -4\kappa \int_{C_x^-} \frac{T_{ij}(x^0 - R, y^a)}{R} dy^1 dy^2 dy^3. \quad (3)$$

R is the Euclidean distance from the point of integration to the point where ψ_{ij} is being evaluated. C_x^- is the retarded Minkowski null cone with vertex x . The right-hand side of equation (3) may be expanded in powers of $1/r$ along the cones $u = \text{const.}$:

$$\psi_{ij} = \frac{-4\kappa}{r} \int_{C_x^-} T_{ij} dx^1 dx^2 dx^3 + O(r^{-2}). \quad (4)$$

The energy-momentum vector P^i is given by the integral

$$P^i = \int_{y^0 = \text{const}} T^{i0} dy^1 dy^2 dy^3.$$

This may be written as an integral over the retarded cone C_x^- with vertex x (Details are given for example in Part II [7]):

$$P^i = \int_{C_x^-} T^{ij} \xi_j dy^1 dy^2 dy^3 + O(r^{-1}).$$

Recall that ξ_i is tangent to C_x^- at the point where it intersects the world-line of the origin. From equation (4) we have

$$\psi_{ij} \xi^j = -\frac{4\kappa}{r} P_i + O(r^{-2}).$$

By our standard choice of time-axis the energy-momentum vector P^i has components

$$P^i = (P^0, 0, 0, 0).$$

Therefore we have

$$\psi_{ij} \xi^i \xi^j = -\frac{4\kappa}{r} P^0 + O(r^{-2}),$$

and from (2)

$$g_{ij}\zeta^i\zeta^j = -\frac{4\kappa}{r}P^0 + O(r^{-2}).$$

We see then that H is given by

$$H = -\kappa P^0. \quad (5)$$

If (1) is radiative then

$$\ddot{h}_{ij} \neq 0.$$

We have therefore a coordinate system in which the components of the metric tensor possess an asymptotic expansion in powers of $1/r$ along a family of forward Minkowski null cones and in which we have

$$H \neq 0$$

and

$$\ddot{h}_{ij} \neq 0.$$

There is however no contradiction with the results of the previous section since they were based on equation (III-9). This equation is quadratic and (3) is valid only in the linear approximation.

The results of Section III assure us however that it is impossible to use (3) as the first approximation in an iterative procedure to obtain a second or higher approximation to an exact retarded solution. For suppose that we have a second approximation $g_{(2)ij}$ to a retarded radiative solution of the field equations. Then in some coordinate system (x^i) , the components of the metric tensor are of the form (III-1). Since

$$|g_{ij}(x) - g_{(2)ij}(x)| = O(\kappa^2)$$

and since the components g_{ij} of the first approximation metric tensor are of the form (III-1) in the harmonic coordinate system (x^i) , (x'^i) may be chosen to differ from (x^i) in the second order only:

$$|x'^i(x) - x^i(x)| = O(\kappa^2).$$

We will then have from equation (III-9)

$$H\ddot{h}_{ij} = 0 \quad (6)$$

where H and $\ddot{h}_{ij}$ are the first order contributions to $H'_{(2)}$ and $\ddot{h}'_{(2)ij}$ respectively. We have dropped the prime since the two coordinate systems (x^i) and (x'^i) are equal in the first approximation. But we saw above that neither H nor $\ddot{h}_{ij}$ vanishes. Therefore (6) is not satisfied.

It is important to remark that it is not the harmonic coordinate condition which we criticize here but rather the use of (3) as a first approximation to a solution. (3) can be considered as a first approximation only in regions of space-time near the sources. This fact was recognized by Fock. He has attempted to find second order approximations to solutions of the field equations using the Schwarzschild metric as zeroth approximation instead of the Minkowski metric [8].

The author wishes to thank M. Papapetrou for a critical reading of the manuscript.

APPENDIX

We show here how the equation (III-7) implies the system of equations (III-8), (III-9). Retaining only the second order terms, we have

$$g_{ij} = \eta_{ij} + \frac{h_{ij}(u, \theta, \phi)}{r} + \frac{h_{ij}^{(2)}(u, \theta, \phi)}{r^2} + O(r^{-3}),$$

$$g^{ij} = \eta^{ij} - \frac{h^{ij}}{r} + O(r^{-2}), \quad h^{ij} = \eta^{ik}\eta^{jl}h_{kl}$$

The Christoffel symbols are given by

$$\begin{aligned} \Gamma_{ij}^k &= \frac{1}{2r} (2\xi_{(i}\dot{h}_{j)}^k - \dot{h}_{ij}\xi^k) + \frac{1}{2r^2} (2\xi_{(i}\dot{h}_{j)}^{(2)k} - \dot{h}_{ij}^{(2)}\xi^k) \\ &+ \frac{1}{2r^2} (2\delta_{(i}^a\dot{h}_{j)a}^k - h_{ij|a}\eta^{ak}) + \frac{1}{2r^2} (2\delta_{(i}^a h_{j)a}^k \xi_a - h_{ij}\delta_a^k \xi^a) \\ &- \frac{1}{2r^2} (2\xi_{(i}\dot{h}_{j)} h^{ak} - \dot{h}_{ij} h^k) + O(r^{-3}). \end{aligned}$$

We have set $h_{ij}\xi^j = h_i$ and for simplicity we have defined

$$h_{i|j|a} = r \left. \frac{\partial h_{ij}}{\partial x^a} \right|_{u=\text{const}}$$

Differentiation gives

$$\begin{aligned} \pi_k &= \frac{1}{2r} (2\xi_{(i}\ddot{h}_{j)}^k \xi_k - \ddot{h}_{ij}\xi^k \xi_k) + \frac{1}{2r^2} (2\xi_{(i}\ddot{h}_{j)}^{(2)k} \xi_k - \ddot{h}_{ij}^{(2)}\xi^k \xi_k) \\ &+ \frac{1}{2r^2} (2\xi_{(i}\dot{h}_{j)}^k - \dot{h}_{ij}\xi^k)\delta_k^a \xi_a \\ &+ \frac{1}{2r^2} (2\pi_{k(i}\dot{h}_{j)}^k - \dot{h}_{ij}\pi_k^i + (2\xi_{(i}\dot{h}_{j)a}^k - \dot{h}_{ij|a}\xi^k)\delta_k^a) \\ &+ \frac{1}{2r^2} (2\xi_{(i}\dot{h}_{j)}^a \xi_a + 2\delta_{(i}^a \dot{h}_{j)a}^k \xi_k - (\dot{h}_{ij|a} + \dot{h}_{ji|a})\eta^{ai})\xi_k \\ &- \frac{1}{2r^2} (2\xi_{(i}\ddot{h}_{j)m} h^{mi} \xi_k - \ddot{h}_{ij} h^i \xi_k) \\ &- \frac{1}{2r^2} (2\xi_{(i}\dot{h}_{j)m} \dot{h}^{mi} \xi_k - \dot{h}_{ij} \dot{h}^i \xi_k) + O(r^{-3}). \end{aligned}$$

We have therefore

$$\begin{aligned} \Gamma_{jk}^i &= \frac{\ddot{h}}{2r} \xi_j \xi_k + \frac{\ddot{h}^{(2)}}{2r^2} \xi_j \xi_k + \frac{\dot{h}}{2r^2} \pi_{jk} + \frac{1}{2r^2} (2\dot{h}\xi_{(j}\delta_{k)}^a \xi_a + 2\xi_{(j}\delta_{k)a}^a \dot{h}_{i|a}) \\ &- \frac{1}{2r^2} (\ddot{h}_{ij} h^{ii} + \dot{h}_{ij} \dot{h}^{ii}) \xi_j \xi_k + O(r^{-3}), \\ \Gamma_{j|i}^i &= \frac{1}{2r} 2\xi_{(j}\ddot{h}_{i)} + \frac{1}{2r^2} \xi_{(j}\ddot{h}_{i)}^{(2)} + \frac{1}{2r^2} (2\xi_{(j}\dot{h}_{i)a}^a \xi_a + 2\xi_{(j}\dot{h}_{i)a}^a \xi_a) \\ &+ \frac{1}{2r^2} (2\delta_{(j}^a \dot{h}_{i)a} + 2\delta_{(j}^a \dot{h}_{i)a}) \xi_a - \frac{1}{2r^2} (2\xi_{(j}\ddot{h}_{i)m} h^{mi} - \ddot{h}_{ij} h(\xi, \xi)) \\ &- \frac{1}{2r^2} (2\xi_{(j}\dot{h}_{i)m} \dot{h}^{mi} - \dot{h}_{ij} \dot{h}(\xi, \xi)) + O(r^{-3}), \end{aligned}$$

$$\Gamma_{ji}^m \Gamma_{mk}^l = \frac{1}{4r^2} (2\dot{h}_j \dot{h}_k + \dot{h}^{(m)} \dot{h}_{lm} \zeta_j \zeta_k) + O(r^{-3}),$$

$$\Gamma_{jk}^m \Gamma_{mi}^l = \frac{1}{2r^2} \dot{h} \zeta_{(j} \dot{h}_{k)} + O(r^{-3}).$$

We have set $h_{ij}^{(2)} \eta^i = h^{(2)}$ and $h_{ij}^{(2)} \zeta^j = h_i^{(2)}$.

The coefficient of $1/r^2$ in the asymptotic expansion of the Ricci tensor is given by

$$\begin{aligned} 2R_{jk}^{(2)} = & \ddot{h}^{(2)} \zeta_j \zeta_k - 2\zeta_{(j} \dot{h}_{k)}^{(2)} + \dot{h} \pi_{jk} + 2(\dot{h} \zeta_{(j} \delta_{k)}^n - \zeta_{(j} \dot{h}_{k)}^n - \delta_{(j}^n \dot{h}_{k)}^n) \zeta_n \\ & + 2(\zeta_{(j} \delta_{k)}^n \dot{h}_{|n} - \zeta_{(j} \dot{h}_{k)|n} - \delta_{(j}^n \dot{h}_{k)|n}) - \left(\ddot{h}_{||} \dot{h}^{||} + \frac{\dot{h}_{||} \dot{h}^{||}}{2} \right) \zeta_j \zeta_k \\ & + 2\zeta_{(j} \dot{h}_{k)m} \dot{h}^m + 2\zeta_{(j} \dot{h}_{k)m} \dot{h}^m + \dot{h}_j \dot{h}_k - \dot{h} \zeta_{(j} \dot{h}_{k)} - \dot{h}_{jk} \dot{h}(\xi, \zeta) - \dot{h}_{jk} \dot{h}(\xi, \zeta). \end{aligned}$$

Using equation (III-5) we have

$$2R_{jk}^{(2)} \zeta^j \zeta^k = 8\dot{H},$$

which gives us

$$\dot{H} = 0.$$

Using this result, we have

$$2R_{jk}^{(2)} \zeta^k = - (4\ddot{H}^{(2)} + \dot{K} + \zeta^k \dot{h}_{k|n}^n - 4\dot{H}_i \dot{H}^i) \zeta_j - 4\dot{H}_j,$$

which gives us

$$\dot{H}_i = 0.$$

Using these two results we have for the trace of $R_{jk}^{(2)}$

$$2R^{(2)} = - 2(4\ddot{H}^{(2)} + \zeta^j \dot{h}_{j|n}^n),$$

which yields us

$$\dot{K} = 0.$$

Using this we have

$$2R_{jk}^{(2)} \zeta^k = - 4(\ddot{H}^{(2)} - \dot{K}) \zeta_j,$$

which gives us

$$\ddot{H}^{(2)} = \dot{K}.$$

Placing these relations in the expression for $R_{jk}^{(2)}$ gives

$$\begin{aligned} 2R_{jk}^{(2)} = & \left(\ddot{K}^{(2)} - 2\dot{h} - 2\dot{h}_{||}^n - \ddot{h}_{||} \dot{h}^{||} - \frac{\dot{h}_{||} \dot{h}^{||}}{2} - 4\dot{K} \right) \zeta_j \zeta_k \\ & - 2(\zeta_{(j} \dot{h}_{k)|n}^n + 2\zeta_{(j} \dot{h}_{k)}^n) \\ & + 4\zeta_{(j} \dot{H}_{k)}^{(2)} - 4\dot{K}_{|n} \zeta_{(j} \zeta_{k)}^n - 4\zeta_{(j} \dot{h}_{k)m} \dot{H}^m - 4\dot{H} \dot{h}_{jk}. \end{aligned}$$

This yields us the rest of the system (III-8), (III-9).

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Gravitational Radiation from a Bounded Source II

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ABSTRACT. — A comparaison is made between the coefficients of $1/r$ in the asymptotic expansion of the components of the metric tensor in a harmonic coordinate system and in a null coordinate system (a coordinate system adapted to a family of forward null hypersurfaces). It is found that the comparison is most simply made if one takes as null coordinate system a radiative coordinate system of the type introduced by Papapetrou.

RÉSUMÉ. — On compare les coefficients de $1/r$ dans le développement asymptotique des composantes du tenseur métrique dans un système de coordonnées harmoniques et un système de coordonnées radiatives du type introduit par Papapetrou.

INTRODUCTION

The purpose of this article is twofold. First of all we wish to give an asymptotic expansion along a family of forward null hypersurfaces $u = \text{const.}$ of the coefficients of the metric tensor in a harmonic coordinate system of a general solution to the Einstein field equations with bounded sources. Secondly, we wish to compare the coefficient of $1/r$ in this expansion with the coefficient of $1/r$ in the asymptotic expansion of the components of the same metric tensor in a radiative coordinate system [1]. That is,

we wish to find the asymptotic expansion of the coordinate transformation which transforms a radiative coordinate system into a harmonic coordinate system.

Define the quantity Γ^i in an arbitrary globally defined coordinate system by

$$\Gamma^i = g^{jk}\Gamma_{jk}^i.$$

What we shall do is investigate the asymptotic transformation properties of Γ^i under two groups of coordinate transformations. The first group, G_1 , is the group of all everywhere regular coordinate transformations which possess asymptotic expansions in powers of $1/r$ of the form (1.7). It is found that Γ_2^i the coefficient of $1/r^2$ of the asymptotic expansion of Γ^i is essentially invariant under G_1 .

A harmonic coordinate system is one in which the quantity Γ^i vanishes. A group G_2 of coordinate transformations is introduced which does not leave Γ_2^i invariant and the coordinate transformation from a radiative coordinate system (x^i) to a harmonic coordinate system (x'^i) is found by finding the element of G_2 which transforms Γ^i to zero.

Consider then a bounded source. By this we mean a body or system of bodies with the property that they are contained in a world-tube of compact space-like extension for all time. Suppose that outside of this world-tube the Ricci tensor vanishes and suppose further that the space-time manifold V_4 is homeomorphic to the four dimensional Euclidian space $\mathbb{R}^4$.

Under these conditions it is reasonable to assume that there exist solutions of the field equations which have the property that in some globally defined coordinate system the components of the metric tensor possess an asymptotic expansion in inverse powers of a radial parameter r along a family of forward null hypersurfaces. Such solutions are called retarded solutions. This assumption can be easily seen to be true in the linear approximation; it has however not yet been proven in the general case.

What one can do is assume a formal power series expansion in powers of $1/r$ along a family of forward null hypersurfaces. The field equations may be then formally integrated. This approach was initiated by Bondi [2] and used by Sachs [3], Newman and Penrose [4] and Papapetrou [1] to study the asymptotic nature of the fields. The success of this method is the main reason for believing in the existence of solutions which possess asymptotic expansions along a family of forward null hypersurfaces. It is found [2], [3] that if coordinate conditions are imposed, the formal integration procedure completely determines the solution to arbitrarily high powers of $1/r$ in terms

of Cauchy data on a specified initial null hypersurface to within two arbitrary functions. These two functions of three variables — the null hypersurface variable u and two angular variables (θ, φ) — describe the amplitudes of the two polarization states which we know a zero-mass particle must have.

In Section I of this article we assume a formal power series expansion in powers of $1/r$ along a family of forward null hypersurfaces for the components of the metric tensor in a harmonic coordinate system. We assume implicitly that there actually exist solutions of the Einstein field equations for the type of sources we are here considering for which the components of the metric tensor in a harmonic coordinate system possess an asymptotic expansion. The only reason we have for believing this to be so is the fact that when a formal expansion is assumed, the field equations may be formally solved up to and including the second order $1/r$ with no obstruction.

In the asymptotic expansion in powers of $1/r$ of the components of the metric tensor in a null coordinate system, the coefficients of $1/r^n$ depend only on the three variables (u, θ, φ) . It is found that the coefficients of $1/r^n$ in the asymptotic expansion of the components of the metric tensor in harmonic coordinate systems depend also on r through the function $\log r$. The coefficient of $1/r^n$ for $n \leq 3$ is of the form

$$g_{ij}(u, r, \theta, \varphi) = p_{ij}^{(n)}(u, \theta, \varphi) + k_{ij}^{(n)}(u, \theta, \varphi) \log r.$$

The necessity of introducing logarithmic terms in an asymptotic expansion of the coefficients of the metric tensor in harmonic coordinates was first noticed by Fock [5].

The harmonic condition does not uniquely define a coordinate system. For example, in the linear approximation the harmonic condition is conserved by transformation of the form

$$x^i \rightarrow x^i + a^i,$$

where a^i is a harmonic vector field. This indeterminacy is reflected in the asymptotic expansion. Instead of finding that the solution depends on the Cauchy data on an initial hypersurface and two arbitrary functions of the variables (u, θ, φ) , one finds that the solution depends on six arbitrary functions of (u, θ, φ) . Two of these functions are the same as those which one finds when using a null coordinate system; the four others reflect the coordinate indeterminacy.

Instead of integrating directly the field equations in a harmonic coordinate system, it is easier to find the coordinate transformation which trans-

forms a given solution in a radiative coordinate system into harmonic coordinates. This is the method which we adopt here. We do not proceed past the second order in the formal integration because of the complexity of the calculations; our interest is only in the coefficient of $1/r$ and how it transforms under the group G_2 of coordinate transformations.

In Section II we shall show that the pseudotensor gives a mass-loss formula which is invariant under the group G_2 of coordinate transformations. In particular it is finite when calculated in a harmonic coordinate system—the $\frac{\log r}{r}$ term does not cause any difficulties. We shall

show also that the expression for the energy-momentum vector given by the pseudotensor in harmonic coordinates is finite in spite of the $\frac{\log r}{r}$ terms. We see then that the $\frac{\log r}{r}$ terms which it is necessary to introduce in an asymptotic expansion of the components of the metric tensor in harmonic coordinates do not give an infinite value either for the mass-loss or for the total energy-momentum vector.

In Section III we give an expression for the Bondi news function and derive an asymptotic expression for the function σ introduced by Newman and Penrose [4], in terms of the components of the metric tensor in harmonic coordinates.

1

Let V_4 be a spacetime manifold whose sources are restricted to a bounded region of space. We shall assume that V_4 admits global coordinate systems:

$$V_4 \xrightarrow{\sim} \mathbb{R}^4.$$

We shall designate a coordinate system either by the application φ or by its image $\varphi(x) = x^i$ in $\mathbb{R}^4$. Latin indices take the values (0, 1, 2, 3); greek indices take the values (1, 2, 3). The signature of the metric is -2 . Indices will be always raised and lowered with the standard Minkowski metric η_{ij} unless otherwise indicated.

Let $u = \text{const.}$ be an arbitrary regular family of forward null hypersurfaces defined outside of a bounded region of the sources. Let r be an affine parameter along the bicharacteristics of $u = \text{const.}$ and let (θ, φ) be a polar coordinate system of one of the spheres $u = \text{const.}, r = \text{const.}$ We shall keep the family $u = \text{const.}$ fixed throughout the rest of the discussion.

This means that we do not touch upon the problems concerning the Bondi-Metzner group [2], [7].

We shall assume that there exists an everywhere regular retarded solution of the field equations. We define a retarded solution of the field equations for the type of matter distribution which we are here considering as a solution with the property that the components of the metric tensor in some globally defined coordinate system possess an asymptotic expansion in inverse powers of r along the family of forward null hypersurfaces $u = \text{const.}$ (and therefore along any regular family of forward null hypersurfaces). We assume also that the globally defined coordinate system may be chosen such that in the limit as r tends to infinity the components of the metric tensor approach η_{ij} :

$$\lim_{r \rightarrow \infty} g_{ij} = \eta_{ij}. \quad (1)$$

We defined previously in Part I [6] a retarded solution as a solution with an asymptotic expansion along a family of cones, null with respect to the Minkowski metric of the coordinate system. We then showed that if the solution is radiative the Minkowski cones must become asymptotically tangent to a family of (Riemannian) null hypersurfaces. Since we are here exclusively interested in radiative solutions, the two definitions are equivalent.

We remark that what follows will also be valid for formal power series solutions of the field equations; that is, solutions with the components of the metric tensor of the form

$$g_{ij} = \eta_{ij} + \sum_1^{\infty} \frac{h_{ij}^{(n)}}{r^n}(u, \theta, \varphi) \quad (2)$$

and where the series may or may not converge. Such solutions are obviously retarded in the above sense. The most compelling reason for considering formal power series solutions is the fact that no exact, retarded, radiative solutions are known with the components of the metric tensor given explicitly in closed form in a globally defined coordinate system such that (1) is satisfied.

Consider a given retarded solution of the field equations. It can be either a function solution, which we shall assume to be of class C^3 for sufficiently large r or a formal power series solution of the form (2). Throughout this article we shall keep this solution fixed. We assume that it is radiative; that is, that the $1/r$ term of the asymptotic expansion of the Riemann tensor does not vanish. We have then by assumption that in some globally defined

coordinate system the components of the metric tensor are of the form

$$g_{ij} = \eta_{ij} + \frac{h_{ij}}{r}(u, \theta, \varphi) + \frac{h_{ij}^{(2)}}{r^2}(u, \theta, \varphi) + O(r^{-3}). \quad (3)$$

To alleviate the formulae in what follows we have dropped the superscript on $h_{ij}^{(1)}$. Let Φ_1 be the set of all coordinate systems such that (3) is satisfied. Φ_1 is the set of all globally defined coordinate systems such that the components of the metric tensor for the solution we are considering possess an asymptotic expansion in powers of $1/r$ along the family of forward null hypersurfaces $u = \text{const.}$ We do not specify to what power n of $1/r$ the components may be expanded. We assume only that $n \geq 2$.

Let G_1 be the group of coordinate transformations which preserve the asymptotic form (3). We note the following two important properties of G_1 . First of all, by definition G_1 acts transitively on the set Φ_1 . If $\gamma \in G_1$ then γ is an everywhere regular map of $\mathbb{R}^4$ onto $\mathbb{R}^4$. If $\varphi, \varphi' \in \Phi_1$ then there exists $\gamma \in G_1$ with $\gamma \circ \varphi' = \varphi$.

Secondly γ must possess an asymptotic expansion in inverse powers of r along the hypersurfaces $u = \text{const.}$ Suppose γ to be given by the functions

$$x^i \rightarrow x'^i = x'^i(x^i). \quad (4)$$

Then the induced transformation of the components of the metric tensor is

$$g'_{ij} \rightarrow g_{ij} = \frac{\partial x'^k}{\partial x^i} \frac{\partial x'^l}{\partial x^j} g'_{kl}. \quad (5)$$

Since g_{ij} and g'_{ij} possess asymptotic expansions so also must $\partial x'^i / \partial x^j$:

$$\frac{\partial x'^i}{\partial x^j} = \delta_j^i + \frac{b_j^i}{r}(u, \theta, \varphi) + \frac{d_j^i}{r^2}(u, \theta, \varphi) + O(r^{-3}). \quad (6)$$

One sees immediately by applying the integrability conditions

$$\frac{\partial^2 x'^i}{\partial x^j \partial x^k} = \frac{\partial^2 x'^i}{\partial x^k \partial x^j}$$

to this system of equations (6) that the map (4) must be of the form

$$x'^i = x^i + a^i \log r + b^i(\theta, \varphi) + \frac{c^i}{r}(u, \theta, \varphi) + \frac{d^i}{r^2}(u, \theta, \varphi) + O(r^{-3}) \quad (7)$$

where the a^i are constants.

From these two elementary properties of the group G_1 we see that to study the asymptotic behaviour of the solution we are considering in an arbitrary coordinate system in Φ_1 it suffices to consider it in a particular coordinate system and then discuss the action of the group G_1 upon the coefficients of the asymptotic expansion. We shall choose as particular coordinate system an element of the class of radiative coordinate systems studied by Papapetrou [1]. These coordinate systems are characterized by the fact that they satisfy the equalities

$$(g_{IJ} - \eta_{IJ})\xi^I = 0, \quad (8)$$

where

$$\xi^I = g^{IJ}\xi_J, \quad \xi_J = \partial u / \partial x^J. \quad (9)$$

We may choose a radiative coordinate system such that for $r > r_0$, r_0 sufficiently large, the family $u = \text{const.}$ coincides with the forward Minkowski null cones whose vertices lie on the world-line of the origin. From the four functions (u, r, θ, φ) construct four functions (x^i) by the equations

$$x^0 = u + r, \quad x^1 = r \sin \theta \cos \varphi, \quad x^2 = r \sin \theta \sin \varphi, \quad x^3 = r \cos \theta, \quad (10)$$

for $r > r_0$. Then (x^i) may be extended to give a globally defined coordinate system

$$\varphi: V_4 \rightarrow \mathbb{R}^4. \quad (11)$$

One can easily show [7] that (x^i) is a radiative coordinate system for $r > r_0$; that is, that the equation (8) holds. One sees also that for $r > r_0$ the hypersurfaces $u = \text{const.}$ are the forward Minkowski null cones of the coordinate system φ whose vertices lie on the world-line of the origin.

We shall choose this coordinate system as our basic coordinate system in Φ_1 . Any other coordinate system in Φ_1 may be obtained from φ by applying an element of G_1 . That is, for all φ' in Φ_1 there exists γ in G_1 such that $\varphi' = \gamma \circ \varphi$.

The fact that the null hypersurfaces along which we shall expand are also Minkowski null cones of the coordinate system φ enables us to use the results of Part I. Consider the asymptotic expansion (3) of the coefficients of the metric tensor in the coordinate system φ . Because of (8) the decomposition of h_{IJ} (See Part I, formula (1.6)) is given by

$$h_{IJ} = \hat{h}_{IJ} + 2\hat{h}_{(I}\xi_{J)} + \frac{h}{2}\pi_{IJ} + \hat{h}\xi_I\xi_J. \quad (12)$$

Condition (8) implies in fact the equations

$$H = H_I = K = 0. \quad (13)$$

Note that, also because of (8), the definition of ξ^i given above (9) coincides with that given in Part I (for $r > r_0$). We showed in Part I that if the solution we are considering is radiative then the quantity $\hat{h}_{ij}$ is necessarily non-zero.

The partial field equations

$$R_{ij} = O(r^{-3}),$$

are equivalent to the following three equations, expressed in terms of the elements of the decompositions of h_{ij} and $h_{ij}^{(2)}$ (See Part I, formulae (III. 8)):

$$\dot{\hat{h}} = 0 \quad (14a)$$

$$\ddot{\hat{h}}^{(2)} - 2\dot{\hat{h}} - 2\dot{\hat{h}}^a{}_{|a} - \ddot{\hat{h}}_{ij}\hat{h}^{ij} - \frac{\dot{\hat{h}}_{ij}}{2}\dot{\hat{h}}^{ij} = 0 \quad (14b)$$

$$\dot{\hat{h}}_j{}^a{}_{|a} + 2\dot{\hat{h}}_j = 0 \quad (14c)$$

A dot designates differentiation with respect to u and a dash, when followed by a greek index, is defined as

$$f_{|a} = r \left. \frac{\partial f}{\partial x^a} \right|_{u=\text{const.}}$$

Papapetrou [I] has further shown that the higher order partial field equations

$$R_{ij} = O(r^{-4}),$$

yield the relation

$$\dot{\hat{h}}^{(2)} = \frac{1}{2} \dot{\hat{h}}_{ij} \dot{\hat{h}}^{ij}. \quad (15)$$

Therefore equation (14 b) may be written as

$$\frac{\partial}{\partial u} \left(2\hat{h} + 2\hat{h}^a{}_{|a} + \frac{\dot{\hat{h}}_{ij}}{2} \hat{h}^{ij} \right) = \frac{1}{2} \dot{\hat{h}}_{ij} \dot{\hat{h}}^{ij}. \quad (16)$$

This may be written as

$$4\dot{M} = -\frac{\dot{\hat{h}}_{ij}}{2} \dot{\hat{h}}^{ij} \quad (17)$$

if we define the function M of (u, θ, φ) as

$$M = -\frac{1}{4} \left(2\hat{h} + 2\hat{h}^a{}_{|a} + \frac{\dot{\hat{h}}_{ij}}{2} \hat{h}^{ij} \right). \quad (18)$$

M is the mass aspect in a radiative coordinate system [I].

We now turn to the problem of finding the elements of the decomposition of the coefficient of $1/r$ in the expansion (3) of the components of the metric tensor in an arbitrary coordinate system in Φ_1 . The transformation (4) from the basic coordinate system φ to a general coordinate system φ' in Φ_1 induces the transformation (5) on the components of the metric tensor. From (7) we see that the coefficients of $1/r$, h_{ij} and h'_{ij} are related as follows:

$$h'_{ij} = h_{ij} + 2a_{(i}\delta_{j)}^a \xi_a - 2\delta_{(i}^a b_{j)a} - 2\dot{c}_{(i}\xi_{j)}. \quad (19)$$

If v_i is any cartesian 4-vector, it may be decomposed as follows:

$$v_i = -\frac{1}{2}(v \cdot \xi)\xi_i - \frac{1}{2}(v \cdot \zeta)\xi_i + \widehat{v}_i, \quad (20)$$

where

$$\widehat{v}_0 = 0, \quad \widehat{v}_a \xi^a = 0.$$

We shall apply this decomposition to a_i and b_i . Decomposing both sides of (19) as we did h_{ij} in formula (I.6) of Part I, we find the following relations between the elements of the decomposition of h_{ij} and those of h'_{ij} :

$$\begin{aligned} H' &= -a \cdot \xi / 2, \\ K' &= \frac{1}{2}(\dot{c} \cdot \xi - a \cdot \xi / 2 - a \cdot \zeta / 2), \\ H'_a &= \frac{1}{2}(\widehat{a}_a - \widehat{b}_a + (\xi \cdot b)_{|a}), \\ \widehat{h}' &= \widehat{h} + \dot{c} \cdot \zeta - a \cdot \zeta / 2, \\ \widehat{K}' &= h - 2\widehat{b}^a_{|a} + 2b \cdot \xi + 2b \cdot \zeta, \\ \widehat{h}'_a &= \widehat{h}_a + \frac{1}{2}(\widehat{a}_a - \widehat{b}_a + (\zeta \cdot b)_{|a} - 2\dot{c}_a), \\ \widehat{h}'_{\alpha\beta} &= \widehat{h}_{\alpha\beta} - 2\widehat{b}_{(\alpha,\beta)} + \pi_{\alpha\beta}\widehat{b}'_{|\gamma} + 2\widehat{b}_{(\alpha}\xi_{\beta)}. \end{aligned} \quad (21)$$

These complicated transformation laws seem to exclude the possibility of attaching too much physical significance to the elements of the decomposition of h_{ij} . However two simple facts may be noticed. First of all, H' has a very simple dependence on (θ, φ) and is independent of u . It is impossible to have an asymptotic expansion in powers of $1/r$ of the form (3) with H' a general function of the variables (u, θ, φ) .

Secondly, the derivative with respect to u of $\hat{h}_{ij}$ is an invariant:

$$\dot{\hat{h}}_{ij} = \dot{\hat{h}}_{ij}.$$

Recall that $\hat{h}_{0i} = 0$.

Since we shall be discussing harmonic coordinate systems, it is of interest to consider the quantities Γ^i , defined by

$$\Gamma^i = g^{jk} \Gamma'_{jk}$$

where Γ'_{jk} are the components of the affine connection in a certain globally defined coordinate system. The necessary and sufficient condition for a coordinate system to be harmonic is that Γ^i vanish.

Let

$$(x^i) \rightarrow (x'^i)$$

be an arbitrary everywhere regular coordinate transformation. This transformation induces the transformation

$$\Gamma'_{jk} \rightarrow \Gamma'_{jk} = \Gamma'_{mn} \frac{\partial x'^m}{\partial x^j} \frac{\partial x'^n}{\partial x^k} \frac{\partial x^l}{\partial x'^l} + \frac{\partial^2 x'^l}{\partial x^j \partial x^k} \frac{\partial x^l}{\partial x'^l}, \quad (22)$$

of the components of the affine connection. Therefore we find the following transformation for the quantities Γ^i :

$$\Gamma'^i \rightarrow \Gamma^i = \Gamma'^j \frac{\partial x^i}{\partial x'^j} + g^{jk} \frac{\partial^2 x'^i}{\partial x^j \partial x^k} \frac{\partial x^l}{\partial x'^l}. \quad (23)$$

Consider now the basic coordinate system ϕ we have introduced. A short calculation gives the following asymptotic expansion of Γ^i in this coordinate system:

$$\Gamma^i = \frac{1}{2r^2} \left[2(\hat{h}^{ia}{}_{|a} + 2\hat{h}^i) + \left(-\dot{h}^{(2)} + 2\hat{h}^a{}_{|a} + 2\hat{h} + \frac{h}{2} + \hat{h}_{ij} \dot{\hat{h}}^{ij} \right) \xi^i + \frac{h}{2} \xi^i \right] + O(r^{-3}). \quad (24)$$

This formula is immediately obtained from the asymptotic expansion of the functions Γ'_{jk} given in the Appendix to Part I. Using equations (15) and (18), equation (24) may be written as

$$\Gamma^i = -\frac{1}{2r^2} \left[\left(4M - \frac{h}{2} \right) \xi^i - \frac{h}{2} \xi^i - 2(\hat{h}^{ia}{}_{|a} + 2\hat{h}^i) \right] + O(r^{-3}). \quad (25)$$

From equation (7) we can calculate the change in the leading term of

the quantity Γ^i . We find that $g^{jk} \frac{\partial^2 x'^i}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial x'^i}$, the term to be added to $\Gamma'^j \frac{\partial x^i}{\partial x'^j}$ in formula (23) is of the form

$$g^{jk} \frac{\partial^2 x'^i}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial x'^i} = -\frac{1}{r^2} (a^i + \Delta b^i) + O(r^{-3}). \quad (26)$$

We have set

$$\eta^{ab} b^i_{|a\beta} = -\Delta b^i.$$

We therefore find from formula (23) that in a general coordinate system φ' in Φ_1 we have

$$\Gamma'^i = -\frac{1}{2r^2} \left[\left(4M - \frac{h}{2} \right) \zeta^i - \frac{h}{2} \zeta^i - 2(\widehat{h}^i_{|a} + 2\widehat{h}^i) - 2a^i - 2\Delta b^i \right] + O(r^{-3}). \quad (27)$$

Let f be a function of the variables (θ, φ) , that is, a function on the sphere. Then a short calculation yields that the sum

$$f_{|a} + 2f\zeta_a$$

is a divergence of a function on the sphere. Therefore the integral

$$\int_{S^2} (f_{|a} + 2f\zeta_a) d\Omega$$

vanishes. $d\Omega$ is the standard volume element of the sphere. We shall often designate the divergence of a function on the sphere simply as div in what follows. Using this notation, equation (27) may be written as

$$\Gamma'^i = -\frac{1}{2r^2} \left[\left(4M - \frac{h}{2} \right) \zeta^i - \frac{h}{2} \zeta^i - 4\widehat{h}^i - 2a^i + \text{div} \right] + O(r^{-3}). \quad (28)$$

We see therefore that the quantity

$$\Gamma^i_2 = \lim_{\substack{r \rightarrow \infty \\ u = \text{const.}}} r^2 \Gamma^i \quad (29)$$

is invariant to within a divergence under the subgroup G_1^0 of G_1 defined by

$$a^i = 0. \quad (30)$$

We have in fact from (25) and (28) that

$$\Gamma'^i_2 = \Gamma^i_2 + a^i + \text{div}.$$

Papapetrou [1] has shown that there exist choices of the family of hyper-

surfaces $u = \text{const.}$ for which the canonically associated radiative coordinate system given by (10) is such that

$$h = 0. \quad (32)$$

In this case we have from (28) the following expression for Γ_2^i :

$$\Gamma_2^i = -4M\xi^i + 4\hat{h}^i + 2a^i + \text{div}. \quad (33)$$

Differentiate equation (25) with respect to u . The field equations (14 a), (14 c), (16) yield the following asymptotic expansion for $\dot{\Gamma}^i$ in the coordinate system φ :

$$\dot{\Gamma}^i = \frac{1}{2r^2} \dot{\hat{h}}_{jk} \hat{h}^{jk} \xi^i + O(r^{-3}). \quad (34)$$

We saw previously that $\hat{h}_{ij}$ is invariant under the group of coordinate transformations G_1 . Equation (34) therefore yields us the following invariant of the group G_1 :

$$\dot{\Gamma}_2^i = \lim_{\substack{r \rightarrow \infty \\ u = \text{const.}}} r^2 \dot{\Gamma}^i. \quad (35)$$

If the field we are considering is radiative, this invariant is non-zero. In a harmonic coordinate system it must of course vanish. This is perhaps the easiest way of seeing that the harmonic coordinate systems do not belong to the coordinate set Φ_1 . The components of the metric tensor in harmonic coordinates do not possess asymptotic expansions of the form (3) in powers of $1/r$ along the hypersurfaces $u = \text{const.}$

Since the group G_1 does not contain the coordinate transformation from φ to a harmonic coordinate system we must look for a larger group $G_2 \supset G_1$ of transformations. Fock [5] found that the asymptotic expansion of the second approximation to a solution of the field equations in harmonic coordinates contains terms of the form $\frac{\log r}{r}$. We are therefore led to consider the group G_2 of regular transformations

$$(x^i) \rightarrow (x'^i)$$

which possess asymptotic expansions of the following form:

$$x'^i = x^i + \left(a^i + \frac{\beta^i}{r} + \frac{\gamma^i}{r^2} + \frac{\delta^i}{r^3} \right) \log r + b^i + \frac{c^i}{r} + \frac{d^i}{r^2} + \frac{e^i}{r^3} + O\left(\frac{1}{r^{3+\alpha}}\right), \quad (36)$$

$\alpha > 0.$

As before the a^i are constants and the b^i are independent of u . The remaining coefficients are functions of (u, θ, φ) . We set

$$G_2 \circ \Phi_1 = \Phi_2.$$

Differentiating both sides of (36) with respect to (x^i) yields us the following equation:

$$\frac{\partial x'^i}{\partial x^j} = \delta_j^i + a_j' \frac{\log r}{r} + \frac{b_j'}{r} + c_j' \frac{\log r}{r^2} + \frac{d_j'}{r^2} + e_j' \frac{\log r}{r^3} + \frac{f_j'}{r^3} + O\left(\frac{1}{r^{3+a}}\right). \quad (37)$$

The coefficients are given by the equations

$$\begin{aligned} a_j' &= \dot{\beta}^i \xi_{j,i}, \\ b_j' &= -a^i \xi_{a,i} \delta_j^a + b_{|a}^i \delta_j^a + \dot{c}^i \xi_{j,i}, \\ c_j' &= \beta_{|a}^i \delta_j^a + \beta^i \xi_{a,i} \delta_j^a + \dot{\gamma}^i \xi_{j,i}, \\ d_j' &= -\beta^i \xi_{a,i} \delta_j^a + c_{|a}^i \delta_j^a + c^i \xi_{a,i} \delta_j^a + \dot{d}^i \xi_{j,i}, \\ e_j' &= \gamma_{|a}^i \delta_j^a + 2\gamma^i \xi_{a,i} \delta_j^a + \dot{\delta}^i \xi_{j,i}, \\ f_j' &= -\gamma^i \xi_{a,i} \delta_j^a + d_{|a}^i \delta_j^a + 2d^i \xi_{a,i} \delta_j^a + \dot{e}^i \xi_{j,i}. \end{aligned} \quad (38)$$

Differentiating a second time yields the following equation:

$$\frac{\partial^2 x'^i}{\partial x^j \partial x^k} = \dot{a}_{j,k}^i \frac{\log r}{r} + \frac{\dot{b}_j^i}{r} \xi_k + b_{j,k}^i \frac{\log r}{r^2} + \frac{c_{j,k}^i}{r^2} + d_{j,k}^i \frac{\log r}{r^3} + \frac{e_{j,k}^i}{r^3} + O\left(\frac{1}{r^{3+a}}\right). \quad (39)$$

The coefficients are given by the equations

$$\begin{aligned} b_{j,k}^i &= a_{j|a}^i \delta_k^a + a_j^i \xi_{a,i} \delta_k^a + \dot{c}_{j,k}^i \xi_k, \\ c_{j,k}^i &= -a_j^i \xi_{a,i} \delta_k^a + b_{j|a}^i \delta_k^a + b_j^i \xi_{a,i} \delta_k^a + \dot{d}_{j,k}^i \xi_k, \\ d_{j,k}^i &= c_{j|a}^i \delta_k^a + 2c_j^i \xi_{a,i} \delta_k^a + \dot{e}_{j,k}^i \xi_k, \\ e_{j,k}^i &= -c_j^i \xi_{a,i} \delta_k^a + d_{j|a}^i \delta_k^a + 2d_j^i \xi_{a,i} \delta_k^a + \dot{f}_{j,k}^i \xi_k. \end{aligned} \quad (40)$$

The contravariant components of the metric tensor in the coordinate system φ are given by

$$g^{jk} = \eta^{jk} - \frac{h^{jk}}{r} + O(r^{-2}). \quad (41)$$

From (39) and (41) we find the following expression for $g^{jk} \frac{\partial^2 x'^i}{\partial x^j \partial x^k}$:

$$\begin{aligned} g^{jk} \frac{\partial^2 x'^i}{\partial x^j \partial x^k} &= \eta^{jk} b_{j,k}^i \frac{\log r}{r} + \frac{\eta^{jk}}{r^2} c_{j,k}^i + (\eta^{jk} d_{j,k}^i - h^{jk} b_{j,k}^i) \frac{\log r}{r^3} \\ &\quad + (\eta^{jk} e_{j,k}^i - h^{jk} c_{j,k}^i) \frac{1}{r^3} + O\left(\frac{1}{r^{3+a}}\right). \end{aligned} \quad (42)$$

After a short calculation, we find that the coefficient of $\frac{\log r}{r}$ vanishes.

$$\eta^{jk} b'_{jk} = 0. \quad (43)$$

A necessary and sufficient condition for this to be true is that a^i be independent of (θ, φ) .

The inverse of the matrix $\partial x'^i / \partial x^j$ may be calculated from (37):

$$\frac{\partial x^i}{\partial x'^j} = \delta^i_j - a^i_j \frac{\log r}{r} - \frac{b^i_j}{r} + O\left(\frac{\log^2 r}{r^2}\right). \quad (44)$$

From equations (42) and (44) we find the following expression for the term to be added to Γ'^i in the transformation (23) induced by a coordinate transformation in G_2 :

$$\begin{aligned} g^{jk} \frac{\partial^2 x'^i}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial x'^l} &= \frac{\eta^{jk}}{r^2} c'_{jk} + (\eta^{jk} d'_{jk} - h^{jk} b'_{jk} - \eta^{jk} c'_{jk} a'_l) \frac{\log r}{r^3} \\ &+ (\eta^{jk} e'_{jk} - h^{jk} c'_{jk} - \eta^{jk} c'_{jk} b'_l) \frac{1}{r^3} + O\left(\frac{1}{r^{3+a}}\right). \end{aligned} \quad (45)$$

If the coordinate system (x'^i) is to be harmonic then we must have

$$\Gamma'^i = 0. \quad (46)$$

Therefore we see from (23) that the transformation

$$(x^i) \rightarrow (x'^i)$$

must satisfy

$$\Gamma^i = g^{jk} \frac{\partial^2 x'^i}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial x'^l}. \quad (47)$$

Let

$$\Gamma^i = \frac{\Gamma^i_2}{r^2} + \frac{\Gamma^i_3}{r^3} + O(r^{-4}) \quad (48)$$

be the asymptotic expansion of Γ^i . Using (45) and (48), equation (47) yields the following three equations:

$$\eta^{jk} c'_{jk} = \Gamma^i_2, \quad (49a)$$

$$\eta^{jk} d'_{jk} - h^{jk} b'_{jk} - \eta^{jk} c'_{jk} a'_l = 0, \quad (49b)$$

$$\eta^{jk} e'_{jk} - h^{jk} c'_{jk} - \eta^{jk} c'_{jk} b'_l = \Gamma^i_3. \quad (49c)$$

If we calculate the left-hand side of equation (49 a) in terms of the coeffi-

cients of the expansion (36), and if we substitute for Γ_2^i its value as given by equation (25), we find the following equation for a^i , b^i and β^i :

$$-a^i - \Delta b^i + 2\dot{\beta}^i = -\frac{1}{2} \left[\left(4M - \frac{h}{2} \right) \xi^i - \frac{h}{2} \zeta^i - 2(\hat{h}^i{}_{;a} + 2\hat{h}^i) \right]. \quad (50)$$

We have here one (vector) equation for three unknowns. However one of the unknowns b^i , is independent of u and the other, a^i , is a constant. We shall return to the problem of solving this equation later.

Equation (49 b), when expressed in terms of the coefficients of the expansion (36) yields the following equation for $\dot{\gamma}^i$:

$$\dot{\gamma}^i = -\frac{1}{4} (2\Delta\beta^i + h\dot{\beta}^i). \quad (51)$$

Equation (49 c), when expressed in terms of the coefficients of the expansion (36) yields the following equation for $\dot{d}^i$:

$$\dot{d}^i = -\frac{1}{4} (2\Delta c^i + h\dot{c}^i) + \frac{\beta^i}{2} + \omega^i. \quad (52)$$

We have here grouped all terms which may be considered as known in the term ω^i . This includes Γ_3^i , which we have not calculated, as well as terms from the left-hand side of (49 c) which depend only on a^i , b^i , β^i or $\dot{\gamma}^i$. $\dot{\delta}^i$ and $\dot{e}^i$ are determined by higher order equations.

The important thing to notice here is that $\dot{c}^i$ remains undetermined. These four arbitrary functions of three variables (u , θ , φ) constitute the time dependent part of the gauge freedom admitted by the harmonic coordinate systems. (They do not constitute all of the gauge freedom. For example, β^i , γ^i , c^i and d^i may be given arbitrary initial values on a certain hypersurface $u = \text{const.}$; their time evolution is then determined by equations (50), (51) and (52).) We shall in what follows suppose for simplicity that the $\dot{c}^i$ vanish. Our purpose here is to discuss the relation between a radiative coordinate system and a harmonic coordinate system. We shall not discuss the relations which exist between two different harmonic coordinate systems.

Consider now the transformation (5) of the components of the metric tensor induced by a coordinate transformation γ

$$(x^i) \rightarrow (x'^i),$$

where $\gamma \in G_2$ and (x^i) is the standard radiative coordinate system. From equation (37) we find that the coefficients of $1/r$ of the asymptotic expansions of the metric tensor in the two coordinate systems are related by the equation

$$h'_{ij} = h_{ij} - 2\eta_{k(i}a_{j)}^k \log r - 2\eta_{k(i}b_{j)}^k. \quad (53)$$

Using equations (38) we find from (53) the equation

$$h'_{ij} = h_{ij} + 2a_{(i}\delta_{j)}^\alpha \xi_\alpha - 2\delta_{(i}^{\alpha\beta} b_{j)}\xi_\alpha - 2c_{(i}\xi_{j)} - 2\dot{\beta}_{(i}\xi_{j)} \log r. \quad (54)$$

This equation may be decomposed as was (19). We find the following relations between the elements of the decompositions of h'_{ij} and those of h_{ij} :

$$\begin{aligned} H' &= -a \cdot \xi / 2, \\ K' &= \frac{1}{2} (\dot{c} \cdot \xi - \dot{\beta} \cdot \xi \log r - a \cdot \xi / 2 - a \cdot \zeta / 2), \\ H'_\alpha &= \frac{1}{2} (\hat{a}_\alpha - \hat{b}_\alpha + (\xi \cdot b)_{|\alpha}), \\ \hat{H}' &= \hat{h} + \dot{c} \cdot \zeta + \dot{\beta} \cdot \zeta \log r - a \cdot \zeta / 2, \\ \hat{K}' &= h - 2\hat{b}^a_{|\alpha} + 2b \cdot \xi + 2b \cdot \zeta, \\ \hat{h}'_\alpha &= \hat{h}_\alpha + \frac{1}{2} (\hat{a}_\alpha - \hat{b}_\alpha + (\xi \cdot b)_{|\alpha} - 2\dot{c}_\alpha - 2\dot{\beta}_\alpha \log r), \\ \hat{h}'_{\alpha\beta} &= \hat{h}_{\alpha\beta} - 2\hat{b}_{(\alpha} \hat{b}_{\beta)} + \pi_{\alpha\beta} \hat{b}^\gamma_{|\gamma} + 2\hat{b}_{(\alpha} \xi_{\beta)}. \end{aligned} \quad (55)$$

We find, as before with the transformations in G_1 , that $\hat{h}'_{ij}$ is an invariant of the group G_2 :

$$\hat{h}'_{ij} = \hat{h}'_{ij}.$$

We see from equations (54) or (55) that h'_{ij} , the coefficient of $1/r$ in the asymptotic expansion of the components of the metric tensor in an arbitrary coordinate system (x^i) in Φ_2 depends on the quantities a^i , b^i , $\dot{c}^i$ and $\dot{\beta}^i$ as well as on the h_{ij} . Suppose that the coordinate system (x^i) is harmonic; suppose, in fact, that it is a particular harmonic coordinate system with

$$\dot{c}^i = 0.$$

This condition determines the harmonic coordinate system to within Cauchy data on a certain initial hypersurface $u = \text{const}$. The quantities h_{ij} may be considered as known. They are given by the equations (14) to within

Cauchy data. The quantities h'_{ij} will then be completely determined by the equations (50) which gives the remaining unknowns a^i , b^i and $\dot{\beta}^i$ in terms of the elements of the decomposition of h_{ij} .

We turn now to the problem of solving equation (50). For this we make two simplifying assumptions. First of all we suppose that the system of bodies we are considering is initially stationary. That is, we assume that there exists a hypersurface

$$u = u_0$$

such that for $u < u_0$ our solution is independent of u . Secondly, we suppose that the family of hypersurfaces $u = \text{const.}$ is such that

$$h = 0.$$

(See formula (32)).

Let M_0 be the value of M for $u < u_0$. Equation (50) may now be written in this region as

$$-a^i - \Delta b^i = (\widehat{h'^{\alpha}}_{| \alpha} + 2\widehat{h}^i) - 2M_0 \xi^i. \quad (56)$$

From the field equations (14c) we see that the first term on the right-hand side of this equation is dependent of u . Therefore, since M_0 is also independent of u , a solution found for $u < u_0$ will remain a solution for $u \geq u_0$. We split equation (56) into the $i = 0$ and $i = \alpha$ components:

$$\Delta b^0 = 2M_0 - a^0, \quad (57a)$$

$$\Delta b^\alpha = -(\widehat{h}\xi^\alpha + \widehat{h}^\alpha + a^\alpha) = (\widehat{h^{\alpha\beta}} + \xi^\alpha \widehat{h}^\beta)_{| \beta}. \quad (57b)$$

We have replaced M_0 in equation (57b) by its defining expression (18) in terms of h_{ij} in the region $u < u_0$:

$$M_0 = -\frac{1}{2}(\widehat{h} + \widehat{h}^\alpha_{| \alpha}).$$

In order for the two equations (57) to have a solution it is necessary that the integral over the sphere of the right-hand side vanish. This condition determines a^i . We have

$$a^0 = \frac{1}{2\pi} \int_{S^2} M_0 d\Omega, \quad (58a)$$

$$a^\alpha = -\frac{1}{4\pi} \int_{S^2} (\widehat{h}\xi^\alpha + \widehat{h}^\alpha) d\Omega. \quad (58b)$$

With these two conditions satisfied we have a solution to equation (57),

unique to within an additive constant. We choose b^i to be the unique solution whose integral over the sphere vanishes.

Consider now the region $u \geq u_0$. From equations (50) with $h = 0$ and (56) we find the following equation for β^i :

$$\dot{\beta}^i = -\delta M \xi^i, \quad (59)$$

where we have set

$$M - M_0 = \delta M.$$

From equation (17) and the definition of M_0 , we see that δM is given by the equation

$$\delta M = -\frac{1}{8} \int_{u_0}^u \hat{h}_{ij} \hat{h}^{ij} du. \quad (60)$$

δM is the mass loss as given for example in [5], [8].

We have now completely determined the quantities h'_{ij} . We give again equation (54), putting in the value of β^i as given above by formula (59):

$$h'_{ij} = h_{ij} + 2a_{(i} \delta_{j)} \xi_a - 2\delta_{(i} b_{j)a} + 2\delta M \log r \xi_i \xi_j. \quad (61)$$

Since a^i and b^i are independent of u , by differentiating equation (61) with respect to u we find from equation (17) the following simple relation between $\dot{h}'_{ij}$ and $\dot{h}_{ij}$:

$$\dot{h}'_{ij} = \dot{h}_{ij} + \frac{1}{4} \hat{h}_{kl} \hat{h}^{kl} \log r \xi_i \xi_j. \quad (62)$$

It is interesting to note that for the initially stationary case we are considering the decomposition

$$M = M_0 + \delta M \quad (63)$$

has physical significance for all values of u . It is, in fact, the second term which determines the logarithmic behaviour of the components of the metric in a harmonic coordinate system. This point was stressed by Isaacson and Winicour [8].

It is also to be noted that the functions β^i are unbounded as functions of u . It is easily seen from equation (59) that β^i tends to infinity with u . For example if the field is radiative for only a finite interval of time, say for

$$u_0 < u < u_1,$$

then from equation (17) we see that for $u > u_1$,

$$M = M_1$$

is independent of u . Set

$$\delta M_{\max} = M_1 - M_0 < 0.$$

Then for large u we have

$$\beta^i \sim -u \delta M_{\max} \xi^i. \quad (64)$$

We may use the asymptotic expansion of an exact solution g'_{ij} of the field equations in a harmonic coordinate system (x'^i) to give an estimate of the distance from the source up to which a solution $g'_{(1)ij}$ of the linearized equations in this coordinate system is a valid first approximation.

From equation (61) we see that a retarded exact solution of the field equations in harmonic coordinates is asymptotically of the form

$$g'_{ij} = \eta_{ij} + \frac{p_{ij}}{r}(u, \theta, \varphi) + 2\delta M(u, \theta, \varphi) \frac{\log r}{r} \xi_i \xi_j + O\left(\frac{1}{r^{1+\alpha}}\right), \quad \alpha > 0. \quad (65)$$

The retarded solution of the linearized equations possess an asymptotic expansion along a family of forward Minkowski null cones $u' = \text{const.}$ of the coordinate system (x'^i) :

$$g'_{(1)ij} = \eta_{ij} + \frac{h_{(1)ij}}{r'}(u', \theta', \varphi') + O(r'^{-2}). \quad (66)$$

(r', θ', φ') is the standard polar coordinate system in the harmonic coordinate space.

Neglecting terms which remain finite at infinity, we find from formula (II-9) that (u, r, θ, φ) and $(u', r', \theta', \varphi')$ are related as follows:

$$\begin{aligned} u &= u' - a^0 \log r' \\ r &= r' \\ \theta &= \theta' \\ \varphi &= \varphi' \end{aligned}$$

(The time-axis is chosen such that $a^0 = 0$). To compare (65) and (66) we give the former as an asymptotic expansion along the hypersurfaces $u' = \text{const.}$:

$$\begin{aligned} g'_{ij} &= \eta_{ij} + \frac{p_{ij}}{r} - a^0 \dot{p}_{ij} \frac{\log r}{r} - \frac{(a^0)^2}{2} \ddot{p}_{ij} \frac{\log^2 r}{r} + \dots \\ &\quad + 2\delta M \xi_i \xi_j \frac{\log r}{r} - 2a^0 \dot{M} \xi_i \xi_j \frac{\log^2 r}{r} + \dots + O\left(\frac{1}{r^{1+\alpha}}\right). \end{aligned} \quad (67)$$

The dots designate higher order terms in the Taylor expansion.

One sees that the two $\frac{\log r}{r}$ terms cannot cancel if $\hat{h}_{ij} \neq 0$, that is, if the field is radiative.

Choose a length l such that $l |g'_{(1)ij,k}|$ and $|g'_{(1)ij}|$ are of the same order of magnitude:

$$l |g'_{(1)ij,k}| \sim |g'_{(1)ij}|,$$

and fix a unit of length such that $l = 1$. Let m be the total mass of the system, v a typical velocity and k the constant of gravitation. Then we have

$$a^0 \sim km \quad \delta M \sim (kmv)^2/v. \quad (68)$$

If (66) is to be a first approximation to an exact solution then we must have (66) and (67) equal to within terms second order in km . This can only be achieved for distances r such that

$$\begin{aligned} |2\delta M \xi_i \xi_j \log r - a^0 \dot{p}_{ij} \log r| &\lesssim mk, \\ |2a^0 \dot{M} \xi_i \xi_j \log^2 r + (a^0)^2 \ddot{p}_{ij} \log^2 r| &\lesssim mk, \end{aligned}$$

and such that the corresponding inequalities for the terms included in the dots on the right-hand side of (67) are satisfied. That is, using (68), for distances such that

$$r \gtrsim e^{1/kmv}.$$

II

We shall now use the Einstein pseudotensor to derive an expression for the total energy-momentum vector in harmonic coordinates and for the mass loss in an arbitrary coordinate system in Φ_2 . Our purpose is to show that the $\frac{\log r}{r}$ terms in the asymptotic expansion of the components of the metric tensor do not give infinite values for these two quantities.

Let φ be an arbitrary globally defined coordinate system and let T_j^i be the mixed components of the matter tensor in this coordinate system. The Einstein energy-momentum pseudotensor is defined as a matrix solution t_j^i to the equation

$$[\sqrt{-g}(T_j^i + t_j^i)]_{,i} = 0. \quad (1)$$

An expression for t_j^i in terms of the components of the metric tensor is to be found for example in Goldberg [9].

A conservation law is obtained by integrating (1) over a certain region

of Space-Time with a regular boundary and applying Green's theorem. The region we shall choose will have in part a null hypersurface as boundary. Green's theorem can not be applied to such regions and we must use Stoke's theorem. We therefore reformulate equation (1) in a form suitable for the application of Stoke's theorem.

We set

$$\hat{T}_{ijkl} = \sqrt{-g} (T_l^m + t_l^m) \epsilon_{mjkl}, \quad (2)$$

and we define the vector 3-form α_i in the coordinate space by the equation

$$\alpha_i = \frac{1}{3!} \hat{T}_{ijkl} dx^j \wedge dx^k \wedge dx^l. \quad (3)$$

One sees immediately that equation (1) is equivalent to the equation

$$d\alpha_i = 0. \quad (4)$$

To define an energy-momentum vector P_i in coordinate space we must integrate the 3-form α_i over a 3-dimensional hypersurface σ . The hypersurface we shall choose is the union of two hypersurfaces $\sigma = \alpha \cup \beta$ where α is a space-like hypersurface and contains the intersection of σ with the matter tube and β coincides with $u = \text{const.}$ at large distances from the matter. Since the form α_i is closed the details of σ near the sources are not important; it is the asymptotic behaviour of σ which determines P_i . We define the energy-momentum vector in coordinate space as the integral of the form α_i over σ :

$$P_i(u) = \int_{\sigma} \alpha_i. \quad (5)$$

We are interested here primarily in fields which are initially stationary and in coordinate systems φ in Φ_2 . If we define for each coordinate system in Φ_2 the vector field P by

$$P(u) = P_i dx^i$$

then these vector fields have a common limit as r tends to infinity along the hypersurfaces $u = \text{const.}$ We may define the total energy-momentum vector as this limit.

To find the equations of motion of P_i we must find the asymptotic value of t_i^j . In any coordinate system in Φ_2 we have, using (I-55), the following expansion for the pseudotensor [10]:

$$t_i^j = \frac{1}{32\pi r^2} \dot{\hat{h}}_{lm} \dot{\hat{h}}^{lm} \xi_i \xi^j + O\left(\frac{1}{r^{2+\alpha}}\right), \quad \alpha > 0. \quad (6)$$

The leading term in the asymptotic expansion of t_j^i is therefore invariant under G_2 . From equations (4), (5) and (6) and using Stoke's theorem, we obtain the following equation:

$$\dot{P}_i = -\frac{1}{32\pi} \int_{S^2} \hat{h}_{lm} \hat{h}^{lm} \xi_i d\Omega. \quad (7)$$

This is the equation of motion for P_i .

We shall now give an expression for the energy-momentum vector P_i in terms of the coefficient of $1/r$ in the asymptotic expansion of the components of the metric tensor. For this we restrict our considerations to the harmonic coordinate system (x'^i) introduced in Section I.

As we remarked previously, P_i depends only on the asymptotic form of the hypersurface σ . This fact may be used to deform σ into a hypersurface σ' which is more convenient for calculations which are carried out in harmonic coordinates. We now proceed with the construction of σ' . We introduced in Section I four functions (u, r, θ, φ) and from them constructed the standard radiative coordinate system (x^i) by the equations (I-10). We use the inverse of equations (I-10) to define four functions $(u', r', \theta', \varphi')$ from the harmonic coordinates (x'^i) . We have then the following diagram:

$$\begin{array}{ccc} (u, r, \theta, \varphi) & \xrightarrow{(I-10)} & (x^i) \\ \downarrow & & \downarrow (I-36) \\ (u', r', \theta', \varphi') & \xleftarrow{(I-10)} & (x'^i) \end{array}$$

The numbers beside the arrows indicate the formulae which define the applications. The diagram defines an application

$$(u, r, \theta, \varphi) \rightarrow (u', r', \theta', \varphi'). \quad (8)$$

To proceed it is convenient to choose a particular orientation for the time axes in the harmonic coordinate system (x'^i) by a Lorentz transformation. We shall choose the time axes parallel to the constant vector a^i . a^i is then given by $a^i = (a^0, 0, 0, 0)$. We also completely determine β^i by choosing $\beta^i = 0$ for $u < u_0$. We have then from (I-59) that $\beta^i = \beta^0 \xi^i$ for all values of u . With these choices it is easily seen that the application (8) is asymptotically of the form

$$\begin{aligned} u' &= u + a^0 \log r + b \cdot \xi - \frac{b_a b^a}{2r} + O\left(\frac{1}{r^{1+a}}\right), \quad a > 0, \\ r' &= r - b^a \xi_a + \beta^0 \frac{\log r}{r} + O(r^{-1}), \\ \theta' &= \theta + O(r^{-1}), \\ \varphi' &= \varphi + O(r^{-1}). \end{aligned} \quad (9)$$

Define $b'^i(\theta', \varphi')$ by

$$b'^i(\theta', \varphi') = b^i(\theta, \varphi).$$

One sees from (9) that

$$b'^i - b^i = O(r^{-1}).$$

We have also

$$\xi_i - \frac{\partial u'}{\partial x'^i} = O(r^{-1}).$$

Therefore we have from (9) that the hypersurface

$$u' - a^0 \log r' - b'^i \frac{\partial u'}{\partial x'^i} + \frac{b'_a b'^a}{2r'} = \text{const.} \quad (10)$$

is asymptotically tangent to the hypersurface σ . We shall choose the hypersurface σ' such that for large r' it is given by (10) and such that for all r' it may be given by an equation of the form

$$x'^0 - x'^0(x'^1, x'^2, x'^3) = \text{const.} \quad (11)$$

The normal p'_i to σ' is given by

$$p'_i = \left(1, -\frac{\partial x'^0}{\partial x'^a} \right). \quad (12)$$

It is independent of x'^0 and satisfies

$$p'_i - \xi'_i = O\left(\frac{1}{r^{1+a}}\right). \quad (13)$$

Recall that ξ'_i are the components of the normal to the hypersurface $u = \text{const.}$ in the coordinate system (x'^i) . In the coordinate system (x'^i) the 3-form α'_i is

$$\alpha'_i = \frac{1}{3!} \sqrt{-g'} (T_i^m + t_i^m) \epsilon_{mjk} dx'^j \wedge dx'^k \wedge dx'^l.$$

Restricted to the hypersurface σ' this equation may be rewritten using (11) as

$$\alpha'_i = \sqrt{-g'} (T_i^m + t_i^m) p'_m dx'^1 \wedge dx'^2 \wedge dx'^3. \quad (14)$$

The 3-form α'_i may be written in terms of the super-potential [9]:

$$\alpha'_i = \frac{1}{16\pi} U_i{}^{jk}{}_{,k} p_j dx'^1 \wedge dx'^2 \wedge dx'^3. \quad (15)$$

Since the super-potential is antisymmetric in the indices (j, k) and since the vector components p'_j satisfy the relation $p'_{[j, k]} = 0$ we may write (15) as

$$\alpha'_i = \frac{1}{16\pi} (U_i'^{jk} p'_j)_{,k} dx'^1 \wedge dx'^2 \wedge dx'^3. \quad (16)$$

We could have proceeded up to this point with the form α'_i restricted to the original hypersurface σ . To proceed we must use the fact that the p'_j are independent of x'^0 ; it was to have this property that we changed the hypersurface of integration. We may consider the expression $U_i'^{jk} p'_j$ as a function of $(\sigma', x'^1, x'^2, x'^3)$. We have therefore

$$(U_i'^{jk} p'_j)_{,k} = \frac{\partial}{\partial \sigma'} (U_i'^{jk} p'_j) p'_k + (U_i'^{ja} p'_j)_{,a}. \quad (17)$$

Since p'_j is independent of x'^0 we have

$$\frac{\partial}{\partial \sigma'} (U_i'^{jk} p'_j) p'_k = \frac{\partial U_i'^{jk}}{\partial \sigma'} p'_j p'_k = 0.$$

Equation (16) may therefore be written as

$$\alpha'_i = \frac{1}{16\pi} (U_i'^{ja} p'_j)_{,a} dx'^1 \wedge dx'^2 \wedge dx'^3. \quad (18)$$

P_i is given as the limit as r'_0 tends to infinity of the integral

$$\int_{\substack{\sigma' \\ r' \leq r'_0}} \alpha'_i.$$

Applying the 3-dimensional Green's theorem for ordinary Euclidean space with a signature -3 we obtain

$$P_i = \lim_{r'_0 \rightarrow \infty} - \frac{1}{16\pi} \int_{S_{r'_0}^2} U_i'^{ja} p'_j n'_a d\Omega. \quad (19)$$

$S_{r'_0}^2$ is the sphere of radius r'_0 and n'_a is its unit normal. Because of (13) we may write (19) as

$$P_i = \lim_{r'_0 \rightarrow \infty} - \frac{1}{16\pi} \int_{S^2} r'^2 U_i'^{ja} \xi'_j n'_a d\Omega.$$

Define A_i by

$$A_i = \lim_{r'_0 \rightarrow \infty} (r'^2 U_i'^{ja} \xi'_j n'_a)$$

P_i is then integral of $-A_i/16\pi$ over the sphere S^2 . Define A_{ij} and B_{ij} by the following expansion of $g'^{ij} = \sqrt{-g'}g'^{ij}$:

$$g'^{ij} = \eta^{ij} + \frac{A^{ij}}{r}(u, \theta, \varphi) + B^{ij}(u, \theta, \varphi) \frac{\log r}{r} + o\left(\frac{1}{r^{1+a}}\right).$$

We are interested in finding A_i to within a divergence on the sphere in terms of A^{ij} and B^{ij} . We remarked in Section I that for any function f of (u, θ, φ) , the expression $f|_a + 2f\xi_a$ is a divergence on the sphere. Using this remark we find, after a rather long but straightforward calculation, the following expression for $A^i (= \eta^{ij}A_j)$:

$$A^i = - (A^{ij}\xi_j + 2B^{ia}\xi_a) + \text{div.} \quad (22)$$

The total energy-momentum vector is given therefore by the equation

$$P^i = \frac{1}{16\pi} \int_{S^2} (A^{ij}\xi_j + 2B^{ia}\xi_a) d\Omega. \quad (23)$$

It is now a simple matter to express P^i in terms of the components of the metric tensor in the radiative coordinate system (x^i) . Using (I-61) we find for example an expression for P^0 as an integral of the quantity M :

$$P^0 = \frac{1}{4\pi} \int_{S^2} M d\Omega. \quad (24)$$

III

We shall now establish the connection between the Bondi news function and the functions $\hat{h}_{ij}$ defined for any coordinate system in Φ_2 by equations (I-12) and (I-55).

Let $(\tilde{x}^i) = (u, r, \theta, \varphi)$ be a Bondi type coordinate system and let (x^i) be the coordinate system obtained therefrom by the coordinate transformation (I-10). Let

$$\tilde{\eta}_{ij} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{bmatrix}$$

be the Minkowski metric in the $(\tilde{x}^i)$ coordinate system. Let λ_a designate a 2×2 matrix of functions which vanish as r^{-a} as r tends to infinity along

the hypersurfaces $u = \text{const.}$ Then if we expand the components of the metric in the $(\tilde{x}^i)$ coordinate system as follows:

$$\tilde{g}_{IJ} = \tilde{\eta}_{IJ} + \frac{\tilde{h}_{IJ}}{r} + \begin{bmatrix} \lambda_2 & \lambda_1 \\ \lambda_1 & \lambda_0 \end{bmatrix}, \quad (1)$$

we find from the results of [3] that $\tilde{h}_{IJ}$ is given by

$$\tilde{h}_{IJ} = \begin{bmatrix} -2M & 0 & r^3 U^2 & r^3 U^3 \\ 0 & 0 & 0 & 0 \\ r^3 U^2 & 0 & -r^2(c + \bar{c}) & -ir^2 \sin \theta (\bar{c} - c) \\ r^3 U^3 & 0 & -ir^2 \sin \theta (\bar{c} - c) & r^2 \sin^2 \theta (c + \bar{c}) \end{bmatrix}, \quad (2)$$

where U^2, U^3 are given by the field equations in terms of c and the initial values. $\bar{c}$ is the Bondi news function.

The components of the metric in the two coordinate systems are related by

$$g_{IJ} = \frac{\partial \tilde{x}^k}{\partial x^I} \frac{\partial \tilde{x}^l}{\partial x^J} \tilde{g}_{kl}. \quad (3)$$

Since we have

$$\eta_{IJ} = \frac{\partial \tilde{x}^k}{\partial x^I} \frac{\partial \tilde{x}^l}{\partial x^J} \tilde{\eta}_{kl}, \quad (4)$$

we find that $\tilde{h}_{IJ}$ and h_{IJ} are related as follows:

$$h_{IJ} = \frac{\partial \tilde{x}^k}{\partial x^I} \frac{\partial \tilde{x}^l}{\partial x^J} \tilde{h}_{kl}. \quad (5)$$

From this relation we calculate that in the coordinate system (x^I)

$$h_{IJ} \epsilon^J = h = 0. \quad (6)$$

Therefore we have

$$h_{IJ} h^{IJ} = \widehat{h}^{IJ} \widehat{h}_{IJ}. \quad (7)$$

If we let $\tilde{\eta}^{IJ}$ be the inverse matrix to $\tilde{\eta}_{IJ}$ (This is an exception to our general rule of lowering and raising indices with η_{IJ}) we find

$$\dot{\widehat{h}}^{IJ} \dot{\widehat{h}}_{IJ} = \dot{h}^{IJ} \dot{h}_{IJ} = \tilde{\eta}^{kl} \tilde{\eta}^{IJ} \dot{\tilde{h}}_{kl} \dot{\tilde{h}}_{IJ} = 8 |\dot{c}|^2. \quad (8)$$

This is the relation between the norm of the Bondi news function and the functions $\widehat{h}_{IJ}$. In Section I we noted that $\widehat{h}_{IJ}$ is invariant under the group G_2 .

Therefore the norm of the Bondi news function is given by the left-hand side of (8) in any coordinate system in Φ_2 .

In conclusion we give the coefficient of $1/r$ in the asymptotic expansion of the function σ introduced by Newman and Penrose [4], in terms of the functions $\hat{h}_{ij}$. Let $(\tilde{x}^i) = (u, r, \theta, \varphi)$ be a Newman-Penrose type coordinate system. We use the same symbol as for the Bondi type coordinate system since to within terms of order $1/r^2$ the two types are the same. We recall from [4] that σ is given by

$$\sigma = l_i m^i m^j. \quad (9)$$

The vectors l_i, m_i are defined in [4]. The vector l_i is identical to the vector ξ_i introduced above in Section I:

$$l_i = \xi_i. \quad (10)$$

In the coordinate system $(\tilde{x}^i)$, the components of l_i are given by

$$\tilde{l}_i = (1, 0, 0, 0).$$

Therefore we have

$$\sigma = -\tilde{\Gamma}_{ij} \tilde{l}^i \tilde{m}^j \tilde{m}^j.$$

This may be written as

$$\sigma = \frac{1}{2} \frac{\partial \tilde{g}_{ij}}{\partial r} \tilde{m}^i \tilde{m}^j + O(r^{-3}). \quad (11)$$

We now substitute into (10) the asymptotic expansion (1) of $\tilde{g}_{ij}$ and use the fact that because of the simple dependence of $\tilde{h}_{ij}$ on r we have

$$\frac{\partial \tilde{h}_{ij}}{\partial r} \tilde{m}^i \tilde{m}^j = \frac{2\tilde{h}_{ij}}{r} \tilde{m}^i \tilde{m}^j. \quad (12)$$

The expression (10) for σ may be then written as

$$\sigma = \frac{1}{2r^2} \tilde{h}_{ij} \tilde{m}^i \tilde{m}^j + O(r^{-3}). \quad (13)$$

Introduce as before a coordinate system (x^i) obtained from $(\tilde{x}^i)$ by the transformation (I-10) and define in the coordinate system (x^i) the limit as r tends to infinity of the components of the vector m^i :

$$m_0^i = \lim_{r \rightarrow \infty} m^i.$$

It follows from the results of [4] that this limit exists and is invariant under G_2 . Using then the transformation formula (5), equation (12) may be written as

$$\sigma = \frac{1}{2r^2} h_{ij} m'_0 m'_0 + O(r^{-3}).$$

The coordinate system (x^i) is a radiative coordinate system [1]. Therefore h_{ij} possesses a decomposition of the form (I-12).

Since we have

$$m'_0 l_i = 0,$$

we obtain using (10) the following expression for σ in terms of $\hat{h}_{ij}$:

$$\sigma = \frac{1}{2r^2} \hat{h}_{ij} m'_0 m'_0 + O(r^{-3}). \quad (14)$$

As we noted in Section I, the coefficient of $1/r^2$ on the right-hand side of (14) is not invariant under the coordinate transformations in G_2 . However if we differentiate with respect to u we find an invariant. It is the Bondi news function [8]:

$$\dot{c} = \frac{1}{2} \dot{\hat{h}}_{ij} m'_0 m'_0. \quad (15)$$

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PHYSIQUE MATHÉMATIQUE. — *Le vecteur énergie-impulsion d'un corps dans un champ gravitationnel quelconque.* Note (*) de M. JOHN MADORE, présentée par M. Louis de Broglie.

Considérons un corps quelconque. Nous allons lui associer deux vecteurs P^i et Q_i et un tenseur antisymétrique M^{ij} . P^i et Q_i seront interprétés comme vecteurs impulsion-énergie et M^{ij} comme moment cinétique.

Soit U le tube d'univers du corps et x un point de U . Soit n_i un vecteur unitaire du genre temps en x et $\sigma(n)$ la surface formée par les géodésiques d'origine x qui sont normales à n_i au point x . Soit $\tilde{n}_i$ le champ des vecteurs unitaires normaux à σ . $\tilde{n}_i$ sera identique à n_i au point x .

Pour chaque point x de U et pour chaque n_i , soit $P^i(x, n)$ le vecteur dont les composantes contravariantes dans un système de coordonnées normales d'origine x sont

$$P^i(x, n) = \int_{\sigma(n)} T^{ij} \tilde{n}_j d\sigma,$$

où l'intégration s'étend seulement sur l'intersection de $\sigma(n)$ avec le tube d'univers U du corps considéré.

A l'aide d'un théorème de Brouwer on démontre que pour un point x donné l'application $n^i \rightarrow P^i/|P|$ possède un point fixe. Soit n_{0i} le vecteur unitaire qui lui correspond et posons $P^i(x) = P^i(x, n_0)$. Par construction $P^i(x) = |P| n_{0i}$. Nous allons supposer qu'il n'y a qu'un seul vecteur n_{0i} ayant cette propriété. Il ne semble pas possible de démontrer ceci sans faire davantage d'hypothèses sur le tenseur de matière T^{ij} .

De même, on peut définir en chaque point x de U , un autre vecteur Q_i comme le vecteur dont les composantes covariantes dans un système de coordonnées normales d'origine x sont

$$Q_i(x) = \int_{\sigma} T_{ij} \tilde{n}_j d\sigma.$$

Ici, et dans la suite $\sigma = \sigma(n_0)$. On remarque qu'en général $Q_i \neq |Q| n_{0i}$.

P^i et Q_i sont deux champs de vecteurs définis dans U . Dans la limite de la relativité restreinte chacun d'eux se réduit à l'expression habituelle de l'impulsion-énergie dans cette théorie.

Le moment cinétique est défini en chaque point x de U par $M^{ij} = P^{ij} - P^{ji}$, où P^{ij} est le tenseur dont les composantes dans un système de coordonnées normales sont

$$P^{ij}(x) = \int_{\sigma} x^i T^{jk} \tilde{n}_k d\sigma.$$

En utilisant le théorème de Brouwer mentionné plus haut, on démontre l'existence d'une ligne C du genre temps sur laquelle l'expression $M^{ij}Q_j$

s'annule. En se restreignant aux valeurs des champs P^i , Q_i , M^{ij} le long de C , on obtient deux vecteurs et un tenseur qui dépendent en général du temps.

Si nous voulons que P^i ou Q_i soient des vecteurs impulsion-énergie, il faut que $|P|$ ou $|Q|$ puisse être interprété comme énergie propre. Notons qu'en effet il y a une troisième possibilité donnée par $\sqrt{P^i Q_i}$. Dans la limite de l'espace plat, chacune de ces trois quantités se réduit à l'énergie propre telle qu'elle est définie habituellement en relativité restreinte.

Pour essayer de clarifier cette ambiguïté, nous allons considérer en détail le cas d'un champ statique à symétrie sphérique. Avec les coordonnées habituelles la métrique est donnée par

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$

En dehors du corps on a la solution de Schwarzschild avec $e^{\nu} = e^{-\lambda} = 1 - 2m/r$. A cause de la symétrie on vérifie que C est la ligne d'univers du centre de symétrie et que $n_a{}^i$ est tangent à C . Toujours à cause de la symétrie Q_i est parallèle à $n_a{}^i$. Nous allons calculer $|P|$ et $|Q|$ au point de C correspondant à $t = 0$.

Soit x' un système normal d'origine $t = r = 0$. On vérifie facilement que σ est l'hypersurface $t = 0$. Les seules composantes non nulles de P^i et Q_i se

$$P^0 = \int_{t=0} \frac{\partial x^0}{\partial t} \rho \tilde{n}^0 d\sigma, \quad Q_0 = \int_{t=0} \frac{\partial t}{\partial x^0} \rho \tilde{n}_0 d\sigma,$$

où $\rho = T^0{}_0$, $\tilde{n}^0 = \sqrt{g_{00}} = e^{-\nu/2}$ et $d\sigma = e^{\lambda/2} d\nu$, où $d\nu = r^2 \sin \theta dr d\theta d\varphi$.

Pour calculer $\partial x^0 / \partial t$ il faut examiner la définition de x^0 de plus près. Pour tout point x' dans un voisinage de σ , soit d la distance géodésique de x' à l'origine. Si x' est assez près de σ , on aura $d \neq 0$. $x^0(x')$ est définie par $x^0/d = dt/ds$ au point $s = 0$, dt/ds étant la composante du genre temps de la tangente à la géodésique joignant l'origine à x' . On vérifie que pour le champ statique considéré, $d/ds [e^{\nu} (dt/ds)] = 0$ le long de toute géodésique passant par l'origine. Nous aurons donc $e^{\nu-\nu(0)} dt/ds = x^0/d$. En intégrant on trouve

$$t = \frac{x^0}{d} \int_0^r e^{\nu(0)-\nu} ds = \frac{x^0}{d} \int_0^r e^{\nu(0)-\nu} \sqrt{e^{\lambda} - e^{\nu} \left(\frac{dt}{dr} \right)^2} dr.$$

On déduit de cette relation

$$\left. \frac{dx^0}{dt} \right|_{t=0} = \frac{\int_0^r e^{\lambda} dr}{\int_0^r \frac{e^{\lambda/2} + \nu(0) - \nu}{e^{\lambda/2} + \nu(0) - \nu} dr}.$$

Finalement on trouve

$$|P| = \sqrt{g_{00}} P^0 = \int_{t=0}^t \omega(r) \rho dv,$$

où

$$\omega(r) = \frac{\int_0^r e^{\frac{1}{2} \int_0^r (v-v(0)) dr} dr}{\int_0^r e^{\frac{1}{2} \int_0^r (v(0)-v(1)) dr} dr} \rho \frac{v-v(0)}{2}.$$

Un calcul analogue donne pour le vecteur Q_i

$$|Q| = \sqrt{g^{00}} Q_0 = \int_{t=0}^t \chi(r) \rho dv,$$

où

$$\chi(r) = \frac{\int_0^r e^{\frac{1}{2} \int_0^r (v(0)-v(1)) dr} dr}{\int_0^r e^{\frac{1}{2} \int_0^r v^2 dr} dr} \rho \frac{1-v(0)-v(1)}{2}.$$

On peut montrer que la masse totale m est donnée par

$$m = \int_{t=0}^t \rho dv.$$

Donc, si l'on veut comparer $|P|$ et $|Q|$ à m , il faut estimer les facteurs $\omega(r)$ et $\chi(r)$.

Pour cela nous avons dû nous restreindre à l'approximation linéaire. A cette approximation, les équations de champ deviennent

$$8\pi\rho = \frac{\lambda}{r^2} + \frac{\lambda'}{r}, \quad \alpha = r v' + \lambda.$$

Nous avons utilisé le fait que la pression est du second ordre. Les expressions pour $\omega(r)$ et $\chi(r)$ deviennent

$$\omega(r) = 1 + \frac{\lambda}{2} + \left(\frac{1}{r} \int_0^r (v - v(0)) dr - \frac{v - v(0)}{2} \right)$$

$$\chi(r) = 1 + \frac{\lambda}{2} - \left(\frac{1}{r} \int_0^r (v - v(0)) dr - \frac{v - v(0)}{2} \right).$$

Le terme entre parenthèses ne semble pas avoir d'interprétation physique immédiate. Toutefois on peut démontrer que $\omega(r) - 1 \geq 0$. Donc nous trouvons que $|P| > m$ à l'approximation linéaire pour toute distribution de matière. Il ne semble pas qu'on puisse démontrer la même inégalité pour $|Q|$.

La situation est donc plutôt compliquée pour les quantités $|P|$ et $|Q|$. Mais nous trouvons un résultat simple pour la quantité $\sqrt{P^i Q_i}$. En effet,

à cause de la linéarité, on démontre tout de suite que

$$\sqrt{V^2 Q} = \int_{t=0}^{\infty} \rho \left(1 + \frac{1}{2} (\psi + \chi + \varphi) \right) dv = \int_{t=0}^{\infty} \rho dv = m + \frac{1}{2} \int_{t=0}^{\infty} \lambda \rho dv.$$

Considérons l'expression

$$\int \left(\lambda + \frac{\varphi}{2} \right) \rho dv.$$

En utilisant les équations du champ et intégrant par parties, on trouve

$$\int \left(\lambda + \frac{\varphi}{2} \right) \rho dv = \frac{1}{4} r \lambda (\lambda + \varphi) \Big|_0^{\infty} + \frac{1}{4} \int_0^{\infty} (\lambda - \varphi' r) \lambda dr = 0.$$

On peut utiliser ce résultat pour déduire que

$$\sqrt{V^2 Q} = m + \frac{1}{4} \int_{t=0}^{\infty} \rho \varphi dv.$$

Le dernier terme a une signification physique directe. En effet, on vérifie sans difficulté que $\varphi = -2\phi$, où ϕ est le potentiel gravitationnel newtonien. Donc on déduit que la différence

$$m - \sqrt{V^2 Q} = -\frac{1}{2} \int_{t=0}^{\infty} \rho \phi dv$$

est l'énergie potentielle du corps dans son champ gravitationnel.

D'après les résultats de la discussion de ce cas particulier, la quantité $\sqrt{V^2 Q}$ a une signification très simple : elle est la différence de m , l'énergie totale, et de l'énergie potentielle.

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The Equations of Motion of an Extended Body in General Relativity

par

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ABSTRACT. — The exponential map $\exp$ is used to associate with an arbitrary extended body an energy-momentum vector P^i and an angular momentum tensor M^{ij} . A center-of-mass line C is defined and the equations of evolution of P^i and M^{ij} along C are given.

RÉSUMÉ. — L'application exponentielle $\exp$ est utilisée pour associer à un corps arbitraire étendu un vecteur énergie-impulsion P^i et un tenseur de moment angulaire M^{ij} . Une ligne d'univers de centre de masse C est définie et les équations d'évolution de P^i et M^{ij} le long de C sont données.

INTRODUCTION

Consider U , the world-tube of an isolated material body described by a matter tensor T^{ij} and let $\tilde{U}$ be its convex hull. That is, $\tilde{U}$ is the smallest set which satisfies the following three conditions: (i) $U \subset \tilde{U}$; (ii) if Y is any space-like geodesic and x and y are any two points of Y which belong to $\tilde{U}$, then the segment xy of Y belongs also to $\tilde{U}$; (iii) the boundary $\partial\tilde{U}$ of $\tilde{U}$ is of class C^1 as a hypersurface. Assume that the field equations are satisfied:

$$G_{ij} = -8\pi T_{ij} \text{ in } U$$

and

$$G_{ij} = 0$$

in a region surrounding U ; and that the solution is of class C^3 in the interior of U and of class C^1 in a neighbourhood of ∂U . G_{ij} is the Einstein tensor. Latin indices take the values (0, 1, 2, 3).

We shall define in $\tilde{U}$ an energy-momentum vector field P^i and an angular momentum tensor field M^{ij} and use them to construct a time-like curve C which always lies in $\tilde{U}$ and may be considered as the world-line of the center of mass of the body. We shall then define an energy-momentum vector and an angular momentum tensor as the restriction of these two fields to C .

In Special Relativity one defines the energy-momentum vector P^i , for example, as an integral of $T^{ij}n_j$ over a space-like hypersurface σ where n_i is the normal to σ . From the conservation laws $T^{ij}{}_{;j} = 0$ one finds that P^i does not depend on the hypersurface σ and since the space is flat there is no problem in integrating the vector $T^{ij}n_j$. A straightforward generalization of this definition to General Relativity encounters two main difficulties: how to choose σ and how to define the integral of $T^{ij}n_j$ over it. Also P^i is no longer in general conserved and cannot as in the flat space of Special Relativity be considered as a « free vector ». We have no longer a vector associated with the body but a vector field defined along a curve C .

Let x be any point in $\tilde{U}$. To define $P^i(x)$, we shall choose as hypersurface of integration a geodesic hypersurface through x such that its normal at x is parallel to $P^i(x)$. We must first of all show that such a hypersurface exists. The problem of defining the integral may be then conveniently solved by the use of the exponential map $\exp$ which maps the tangent space at x onto the space-time manifold V_4 . The inverse of this mapping always exists in a neighbourhood of x . We assume this neighbourhood sufficiently large to include all points of σ where T_{ij} does not vanish and define the integral of $T^{ij}n_j$ over σ as the integral of the image of $T^{ij}n_j$ in the tangent space at x over the image of σ . By our choice of σ its image is the linear hypersurface perpendicular to $P^i(x)$. By construction, $P^i(x)$ is an element of the tangent space at x ; that is, it is a vector. We shall construct the angular momentum tensor field M^{ij} similarly.

Apart from the standard restriction on T^{ij} that $T^{ij}u_iu_j > 0$ for all x in U and for all u_i time-like, we make several convenient restrictions on the field strength. These are listed at the end of section I. Solutions excluded by these restrictions would probably be of no interest in macrophysics. We refer to two articles by Dixon [1], [2] for a description of an alternate way of defining P^i and M^{ij} as well as for references to the previous literature.

1

Let T_x^* be the cotangent space at a point x of U and let n_i be a unit time-like vector in T_x^* . Let $\sigma(n)$ be the geodesic hypersurface through x perpendicular to n_i . That is, $\sigma(n)$ is the hypersurface formed by the geodesics through x with tangent at x perpendicular to n_i . Let n'_i denote the unit vector field normal to σ . Therefore

$$n'_i = n_i \text{ at } x.$$

The prime will be dropped when there is no risk of confusion. We shall assume that each point $x \in \tilde{U}$ is the origin of a normal coordinate system defined in a neighbourhood of x which includes the sets $\sigma(n) \cap U$ where n_i varies over the time-like unit vectors in T_x^* . That is, we assume that the exponential map is invertible in this neighbourhood.

At each point $x \in \tilde{U}$ and for each time-like $n_i \in T_x^*$, let $P^i(x; n)$ be the vector whose components in a normal coordinate system with origin x are given by

$$P^i(x; n) = \int_{\sigma(n)} T^{ij} n'_j d\sigma. \quad (1)$$

Here and in the following $d\sigma$ denotes the measure determined by the metric induced on σ .

We assume that all of these vectors are time-like and that they remain bounded away from the light cone as n_i varies over the time-like elements of T_x^* . This is based on the interpretation of $P^i(x; n)$ as the energy-momentum vector of a macroscopic body. We can then conclude by the Brouwer fixed-point theorem that the map $n^i \rightarrow P^i/|P|$ where $|P|$ is the norm of the vector P^i , has a fixed point. We shall assume that this fixed point is unique and designate by $P^i(x)$ the vector at x so obtained. $P^i(x)$ will be called the energy-momentum vector field. In what follows σ will always designate the geodesic hypersurface through x whose normal at x is parallel to $P^i(x)$.

At each point $x \in \tilde{U}$, we define the 2^m -pole moment $P_{(x)}^{i_1 \dots i_m j_1 \dots j_m}$ as the tensor whose components in a normal coordinate system with origin x are given by

$$P(x)^{i_1 \dots i_m j_1 \dots j_m} = \int_{\sigma} x^{i_1} \dots x^{i_m} T^{jk} n'_k d\sigma, \quad (2)$$

where x^i are the coordinates in the normal coordinate system of the point of integration.

The energy-momentum vector and the multipole moment tensors may be also defined in a coordinate-free manner as follows. Let $\alpha \in T_x^*$ and set $x^* = \exp^{*-1}\alpha$. Then the energy-momentum vector $P(x)$ is given by

$$P(x)(x) = \int_{\sigma} T(x^*, n) d\sigma. \quad (3)$$

Let $\alpha_1, \dots, \alpha_{m-1}, \beta \in T_x^*$ and set $\beta^* = \exp^{*-1}\beta$, $X = \exp^{-1}x'$ where x' is the point of integration in σ . Then the 2^m -pole moment is given by

$$P(x)(\alpha_1, \dots, \alpha_m, \beta) = \int_{\sigma} X(\alpha_1) \dots X(\alpha_m) T(\beta^*, n) d\sigma. \quad (4)$$

We shall not discuss this notation further since we do not use it in what follows.

Define the linear combinations of the 2^m -pole moments $M^{i_1 \dots i_m j}$ by

$$M^{i_1 \dots i_m j} = 2P^{i_1 \dots i_m j} - P^{(i_1 \dots i_m j)}, \quad (5)$$

where, for example $P^{(ij)} = P^{ij} + P^{ji}$, M^{ij} is the angular momentum tensor.

We have defined the energy-momentum vector at x to be $P^i(x)$ derived from the contravariant form of the matter tensor. We could equally well have defined it to be $Q_i(x)$ derived from the mixed form. That is, $Q_i(x)$ is the vector whose components in a normal coordinate system with origin x are given by

$$Q_i(x) = \int_{\sigma} T^j_{i} d\sigma. \quad (6)$$

Since in general the components of the metric tensor in a normal coordinate system are not constants, the contravariant form of Q_i will not in general be equal to P^i .

If P^i (or Q_i) is to be the energy-momentum vector, then $|P|$ (or $|Q|$) will have to have an interpretation as the rest energy of the body. We have in fact three possible definitions of energy: $|P|$, $|Q|$ and $\sqrt{g(P, Q)}$ and no *a priori* reason for preferring one over the other two. In the limit of flat space, all three of these quantities reduce to the usual expression for rest energy in Special Relativity. For a discussion of this point in the case of the Schwarzschild solution see reference [3].

We shall now use the vector field P^i and the tensor field M^{ij} which we have defined everywhere in $\tilde{U}$ to construct a time-like curve C which lies in $\tilde{U}$ and which may be interpreted as the world-line of the center of mass.

The energy-momentum vector and the multipole moment tensors may be also defined in a coordinate-free manner as follows. Let $x \in T_x^+$ and set $x^* = \exp^{*-1}x$. Then the energy-momentum vector $P(x)$ is given by

$$P(x)(x) = \int_{\sigma} T(x^*, n) d\sigma. \quad (3)$$

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To this end we consider the inner product $M^{ij}P_j$ at a point $x \in \tilde{U}$. We have in a normal coordinate system with origin x

$$M^{ij}P_j = \int_{\sigma} x^i T^{jk} n_k P_j d\sigma. \quad (7a)$$

Since P_j is parallel to the normal to σ at the point x and since x^i is a vector tangent to σ at x

$$P_i x^i = 0.$$

Therefore we have

$$M^{ij}P_j = \int_{\sigma} x^i T^{jk} n_k P_j d\sigma. \quad (7b)$$

So the vector $M^{ij}P_j$ at the point x is space-like and tangent to the hypersurface σ .

Consider the expression $T^{jk} n_k P_j$ which appears in the integral in (7b). The functions n_k are the components of a vector at a point of integration x' of σ and P_j are the components of a vector at x . Consider the vector at x' whose components in a normal coordinate system with origin x are P_j . Then P_j and n'_j are parallel. In fact, if the normal coordinate system is chosen such that σ is given by $t = 0$, both P_j and n'_j have only the zero component non null. We may therefore conclude that the expression $T^{jk} n_k P_j$ is never negative:

$$T^{jk} n_k P_j = c T^{jk} n'_j n'_k \geq 0,$$

where c is some positive constant.

Consider now $M^{ij}P_j$ at a boundary point $x \in \partial\tilde{U}$. Let l be the geodesic

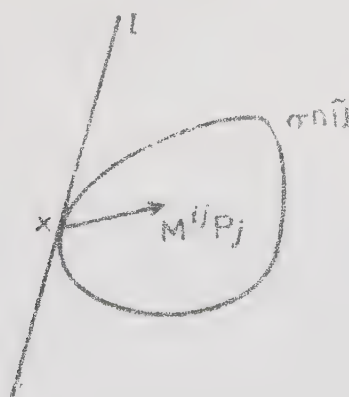


Fig. 1.

plane in σ tangent to $\partial\tilde{U}$ at x . By construction $\tilde{U} \cap \sigma$ lies to one side of L . We see from (7b) that the vector $M^{ij}P_j$ always lies on the same side of L as $\tilde{U} \cap \sigma$ and that it can never lie in L .

From this fact and from the fact that $M^{ij}P_j$ is always space-like, one deduces, again using the Brouwer fixed-point theorem, that any space-like section of $\tilde{U}$ contains a zero of $M^{ij}P_j$. We shall assume that this zero is unique. Beigleböck [4] has shown that this is so, if one places additional restrictions on the matter tensor. Denote by C the time-like curve formed by the zeros of $M^{ij}P_j$. This criteria for the center of mass was introduced by Pryce [5]. Since P^i and M^{ij} are of class C^1 so is C . The curve C will be called the world-line of the center of mass. One sees from (7b) that in the Newtonian limit it reduces in fact to this line.

Apart from the assumptions relative to the positivity of the matter density: that $T^{ij}n_in_j \geq 0$ and that $P^i(x; n)$ is a time-like vector for n_i time-like, we have made three physical assumptions in this section. They may be formulated as follows: if x and y are two points of $\tilde{U}$ which can be joined by a space-like geodesic, then this geodesic is unique; the map $n^i \rightarrow P^i/|P|$ has a unique fixed point; the vector $M^{ij}P_j$ has a unique zero on any space-like section of $\tilde{U}$. In general one could not expect these three assumptions to hold. Each constitutes a restriction on the strength of the field.

II

We shall now derive differential equations for P^i and M^{ij} along the curve C . As we shall never explicitly use the defining relation for C , the same method may be used to derive differential equations for P^i and M^{ij} along any time-like curve in $\tilde{U}$. The basis for these equations is the integrability condition for the Einstein field equations, that is the conservation laws

$$T^{ij}_{;j} = 0. \quad (8)$$

For each point x of C we have the geodesic hypersurface σ through x determined in Section I such that its normal at x is parallel to P^i .

Let t be the geodesic parameter along C , chosen such that $t = 0$ is the point x , and let y be a point of C with $t > 0$. Let σ and σ' be respectively the geodesic hypersurfaces through x and y and V the part of $\tilde{U}$ between them. Choose a normal coordinate system x^i with center x and a normal

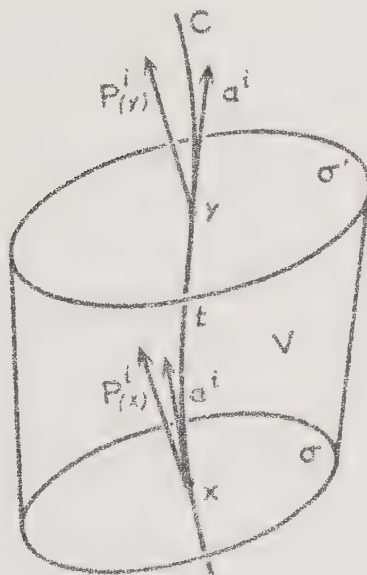


FIG. 2

coordinate system y^i with center y . Choose the basis of y^i the basis of x^i transported parallelly along C to y . This choice of basis at y gives

$$\frac{\partial y^j}{\partial x^i} = \delta^j_i + O(t^2) \text{ on } C,$$

a fact which will be useful later. Denote by a^i the unit tangent vector field to the curve C . Components of tensors with respect to the coordinate system x^i are unprimed; those with respect to the coordinate system y^i are primed. For convenience set

$$\lim_{t \rightarrow 0} \frac{dv}{t} = d\sigma.$$

The coordinate transformation $y^i = f^i(x^j)$ contains t as a parameter. For $t = 0$ we have $f^i(x^j) = x^i$. Define μ^i by

$$y^i = x^i - ta^i - t\mu^i + O(t^2), \quad (9)$$

where a^i are the components of the tangent to C at x in the normal coordinate system with origin x . One sees that μ^i is the part of $-\left.\frac{\partial f^i}{\partial t}\right|_{t=0}$ which

depends on the curvature of the space. In a flat space $y^i = x^i - t\alpha^i$ exactly. Differentiation with respect to x^i gives

$$\frac{\partial y^i}{\partial x^j} = \delta_j^i - t\mu_{,j}^i + O(t^2), \quad (10)$$

with

$$\mu_{,i}^i = 0 \text{ on } C.$$

By definition, we have at y

$$P^i(y) = \int_{\sigma'} T^{ij} n_j d\sigma.$$

Therefore we have

$$\frac{\partial y^i}{\partial x^j} P^j(y) = \int_{\sigma'} \frac{\partial y^i}{\partial x^k} T^{kj} n_j d\sigma. \quad (11)$$

Substituting the above expansion (10) for $\frac{\partial y^i}{\partial x^j}$ gives

$$\int_{\sigma'} T^{ij} n_j d\sigma = P^i(y) + t \int_{\sigma'} \mu_{,k}^i T^{kj} n_j d\sigma + O(t^2). \quad (12)$$

Since the metric has been supposed to be of class $\mathfrak{C}^1$ we have from the joining conditions $T^{ij} v_j = 0$, where v_i is the unit normal to ∂U . The conservation laws (8) may be written as

$$\frac{1}{\sqrt{-g}} (\sqrt{-g} T^{ij})_{,i} + \Gamma_{jk}^i T^{jk} = 0. \quad (13)$$

Integrating this over V and applying Green's theorem to the first term on the left-hand side gives

$$\int_{\sigma'} T^{ij} n_j d\sigma - P^i(x) + \int_V \Gamma_{jk}^i T^{jk} dV = 0. \quad (14)$$

Substituting the expression (12) for $\int_{\sigma'} T^{ij} n_j d\sigma$ in this equation gives

$$P^i(y) - P^i(x) + t \int_{\sigma'} \mu_{,k}^i T^{kj} n_j d\sigma + \int_V \Gamma_{jk}^i T^{jk} dV = O(t^2). \quad (15)$$

Dividing by t and taking the limit $t \rightarrow 0$ gives the following differential equation for P^i :

$$\frac{dP^i}{dt} + \int_{\sigma'} \mu_{,k}^i T^{kj} n_j d\sigma + \int_V \Gamma_{jk}^i T^{jk} d\bar{\sigma} = 0. \quad (16)$$

This differential equation describes the evolution of P^i along the curve C .

An exactly similar procedure leads to a differential equation for M^{ij} . The details are as follows. By definition, we have at y

$$P^{ij}(y) = \int_{\sigma} x^i T^{jk} n_k d\sigma. \quad (17)$$

Therefore

$$\frac{\partial y^i}{\partial \lambda^k} \frac{\partial y^j}{\partial \lambda^l} P^{kl} = \int_{\sigma} x^i \frac{\partial y^j}{\partial \lambda^l} T^{kl} n_k d\sigma. \quad (18)$$

Substituting the expansion of y^i and $\frac{\partial y^i}{\partial \lambda^j}$ in this expression gives

$$\int_{\sigma} x^i T^{jk} n_k d\sigma = P^{ij}(y) + t a^i P^j(y) + t \int_{\sigma} \mu^i T^{jk} n_k d\sigma + t \int_{\sigma} x^i \mu^j_l T^{lk} n_k d\sigma + O(t^2). \quad (19)$$

From (13) we arrive at the identity

$$\frac{1}{\sqrt{-g}} (\sqrt{-g} x^i T^{jk})_{,k} - T^{ij} + x^i \Gamma^j_{kl} T^{kl} = 0. \quad (20)$$

Integrating this over V and applying Green's theorem to the first term on the left-hand side gives

$$\int_{\sigma} x^i T^{jk} n_k d\sigma - P^{ij}(x) - \int_V T^{ij} dV + \int_V x^i \Gamma^j_{kl} T^{kl} dV = 0. \quad (21)$$

Therefore, using (19) we have

$$P^{ij}(y) - P^{ij}(x) + t a^i P^j(y) + t \int_{\sigma} \mu^i T^{jk} n_k d\sigma + t \int_{\sigma} x^i \mu^j_l T^{lk} n_k d\sigma \\ - \int_V T^{ij} dV + \int_V x^i \Gamma^j_{kl} T^{kl} dV = O(t^2). \quad (22)$$

Dividing by t and taking the limit $t \rightarrow 0$ gives the following differential equation for P^{ij} :

$$\frac{dP^{ij}}{dt} + a^i P^j + \int_{\sigma} \mu^i T^{jk} n_k d\sigma + \int_{\sigma} x^i \mu^j_l T^{lk} n_k d\sigma - \int_{\sigma} T^{ij} d\sigma + \int_{\sigma} x^i \Gamma^j_{kl} T^{kl} d\sigma = 0. \quad (23)$$

Taking the antisymmetrical part of this equation yields

$$\frac{dM^{ij}}{dt} + a^{[i} P^{j]} + \int_{\sigma} \mu^{[i} T^{j]k} n_k d\sigma + \int_{\sigma} x^{[i} \mu^{j]}_l T^{lk} n_k d\sigma + \int_{\sigma} x^{[i} \Gamma^{j]}_{kl} T^{kl} d\sigma = 0. \quad (24)$$

This differential equation describes the evolution of M^{ij} along the curve C .

III

In this section we derive a multipole expansion for the various integrals in equations (16) and (24), retaining terms up to the quadrupole order. We shall make constant use of the fact that the Γ_{jk}^i are the components of the affine connection in a normal coordinate system. Hence we have

$$\Gamma_{ijk, \dots, lmn} = 0 \text{ at } x,$$

and the following equations can be immediately seen to be satisfied at x :

$$\Gamma_{jk, n}^i \xi^{jk} = -\frac{2}{3} R_{mn}^i \xi^{jk}, \quad (25a)$$

$$\Gamma_{jk, lmn}^i \xi^{jk} = \frac{1}{3} R_{jnk, l}^i \xi^{jk}, \quad (25b)$$

where ξ^{jk} and ξ^{jkl} are the components of arbitrary completely symmetric tensors.

First of all we calculate a multipole expansion for the integrals involving μ^i and $\mu_{,j}^i$. Let x' be a point in a neighbourhood of x (fig. 3) and let C' be the geodesic joining x' and y with affine parameter S normalized in such a way that y is the point $S = 0$ and x' is the point $S = 1$. For all points x' near enough to σ , t can be chosen small enough that the geodesic is space-like and hence unique by the assumption made in Section I. Let x^i be the coordinates of the point x' in the normal coordinate system with origin x . We should logically designate the coordinates of this point by x'^i but since the coordinates of the point x are $(0, 0, 0, 0)$ we drop the prime to alleviate the formulæ. Since the curve C' is parameterized by S the coordinates $x^i(x')$ of any point x'' on C' may be considered as functions $x^i(S)$ of S . Since the metric has been assumed to be of class C^3 we may expand $x^i(S)$ in a Taylor series expansion about the point y , which is given by $S = 0$. We have therefore

$$x^i(S) = x^i(y) + S \frac{dx^i}{dS}(y) + \frac{S^2}{2!} \frac{d^2 x^i}{dS^2}(y) + \frac{S^3}{3!} \frac{d^3 x^i}{dS^3}(y) + f^i(S), \quad (26a)$$

where $f^i(S)$ is a function which vanishes as S^4 at $S = 0$. In particular for $S = 1$ we have the coordinates of the point x' :

$$x^i = x^i(y) + \frac{dx^i}{dS}(y) + \frac{1}{2!} \frac{d^2 x^i}{dS^2}(y) + \frac{1}{3!} \frac{d^3 x^i}{dS^3}(y) + f^i(1). \quad (26b)$$

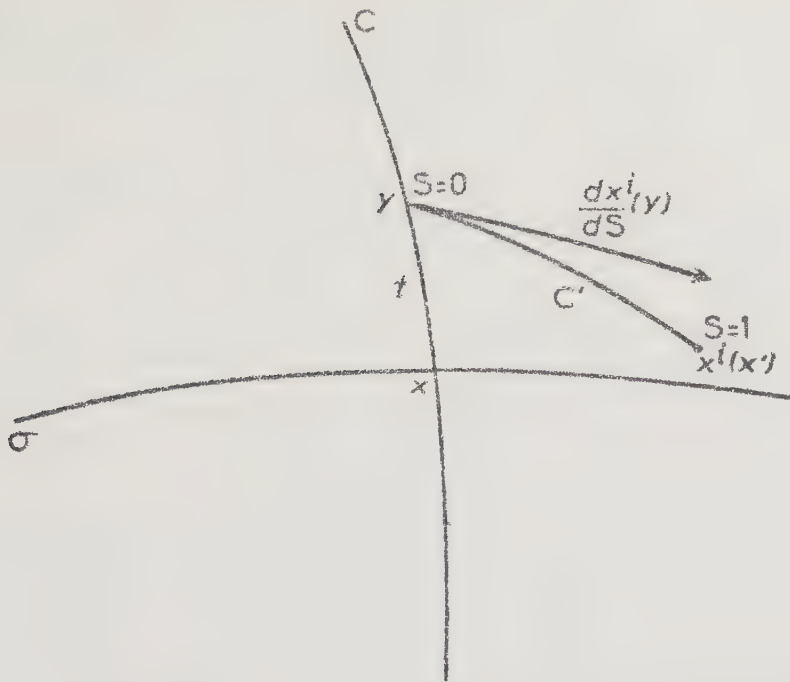


FIG. 3.

The coordinates $x^i(y)$ of the point y are given by

$$x^i(y) = a^i t + O(t^2). \quad (27)$$

By definition the functions $\frac{dx^i}{dS}(y)$, the components of the tangent to the curve C' at the point y , are the coordinates of the point x' in the normal coordinate system with origin y . We have therefore

$$\frac{dx^i}{dS}(y) = y^i.$$

Since C' is a geodesic, we have at the point y

$$\frac{d^2 x^i}{dS^2}(y) = -\Gamma_{jk}^i(y) y^j y^k. \quad (28)$$

But the $\Gamma_{jk}^i(y)$ are the components of the affine connection in a normal coordinate system with origin x . From (27) we have therefore

$$\Gamma_{jk}^i(t) = t \Gamma_{jk}^i(a) + O(t^2)$$

Here and in the rest of this section if the point of valuation of a function is not given it is to be considered as the point x . Since we have

$$y^i = x^i + O(t),$$

equation (28) may be written as

$$\frac{d^2 x^i}{dS^2}(t) = -(\Gamma_{jk}^i x^j x^k a^l + O(t^2),$$

A similar expression for $\frac{d^2 x^i}{dS^2}(y)$ is obtained by differentiating the equation

$$\frac{d^2 x^i}{dS^2} + \Gamma_{jk}^i \frac{dx^j}{dS} \frac{dx^k}{dS} = 0$$

and setting $S = 0$. Substituting these expressions in equation (26b) yields

$$y^i = x^i - ta^i + \frac{t}{2!} \Gamma_{jkn}^i x^j x^k a^n + \frac{t}{3!} \Gamma_{jklm}^i x^j x^k x^l a^m - f^i(t) + O(t^4).$$

The functions $f^i(t)$ are of fourth order in the coordinates x^i and do not contribute to the quadrupole terms. Therefore in the following we shall neglect them.

Using the expressions (25) for the derivatives of the components of the affine connection at the point x one derives from its definition by formula (9), the following expression for μ^i :

$$\mu^i = \frac{2}{3!} R_{jkn}^i x^j x^k a^n + \frac{2}{4!} R_{jklm}^i x^j x^k x^l a^m \quad (29)$$

Using this we see that the second term in equation (16) is given by

$$\int_{\sigma} \mu_{,k}^i \Gamma^{kj} n_j d\sigma = \frac{2}{3!} R_{jkn}^i a^n \Gamma^{jn} + \frac{2}{4!} R_{jklm}^i a^m \Gamma^{jklm}. \quad (30)$$

The sum of the third and fourth terms in equation (24) is given by

$$\int_{\sigma} \mu^{(i} \Gamma^{jk)} n_k d\sigma + \int_{\sigma} \mu_{,i}^{(i} \Gamma^{jk)} n_k d\sigma = \frac{2}{3!} M^{kl[i} R_{klm}^{j]} a^m. \quad (31)$$

Next we calculate a multipole expansion for the integrals involving the components of the affine connection. As with the previous calculations, we here retain only those terms which contribute to the multipoles up to

and including the quadrupole. This means that we neglect all terms containing more than two factors x^i . We set for convenience

$$\omega^{(ij)k} = \int x^{(i} \dots x^{j)} \Gamma^{jk} d\sigma.$$

Expanding Γ'_{jk} about the point x , gives, for the last term in equation (16) the expression

$$\int_{\sigma} \Gamma'_{jk} \Gamma^{jk} d\sigma = \Gamma'_{jk,l} \omega^{lj} + \frac{1}{2} \Gamma'_{jk,lm} \omega^{lmj}, \quad (32)$$

For the last term in equation (24), we obtain the expression

$$\int_{\sigma} x^{kl} \Gamma'_{kl} \Gamma^{kl} d\sigma = \Gamma'_{kl,lmn} \omega^{lmkl}. \quad (33)$$

We shall now find expressions for the terms on the right-hand side of equations (32) and (33) as functions of the linear combinations of multipole moments M^{ij} and M^{ijk} . For this we must use again the conservation laws. We find using (13) the following identity

$$\frac{1}{\sqrt{-g}} (\sqrt{-g} x^i \Gamma^{kj})_{,i} - x^{ij} \Gamma^{jk} + x^i x^j \Gamma'_{lm} \Gamma^{lm} = 0. \quad (34)$$

Integrating this over V , applying Green's theorem to the first term, dividing by t and taking the limit $t \rightarrow 0$ gives the following equation for $\omega^{(ij)k}$:

$$\frac{d\mathbf{p}^{ijk}}{dt} + a^{(i} \mathbf{p}^{jk)} = \omega^{(ij)k} = 0. \quad (35)$$

We do not give in detail the intermediate steps since they are exactly the same as those which lead from equation (20) to equation (23). Since ω^{ijk} is symmetric in its last two indices, it may be expressed in terms of $\omega^{(ij)k}$:

$$\omega^{ijk} = \frac{1}{2} (\omega^{(ij)k} + \omega^{(jk)i} + \omega^{(ki)j}).$$

Therefore we have from (35) the following expression for $\omega^{(ij)k}$:

$$2\omega^{(ij)k} = a^k \mathbf{p}^{(ij)} - a^{(i} \mathbf{M}^{jk)} = \frac{d\mathbf{M}^{ijk}}{dt}. \quad (36)$$

From this we find the following expression for the first term on the right-hand side of (32):

$$\Gamma_{jk,l}^i \omega^{ijk} = \frac{1}{2!} R_{ajk}^i a^n M^{jk} + \frac{2}{3!} R_{jkn}^i \frac{dM^{jkn}}{dt} - \frac{2}{3!} R_{jkn}^i a^n P^{(jk)}, \quad (37)$$

Proceeding in a similar manner, from (13) we obtain the identity

$$\frac{1}{\sqrt{-g}} (\sqrt{-g} x^i x^j x^k T^{lm})_{,m} - x^{(i} x^j T^{k)l} + x^i x^j x^k \Gamma_{mn}^l T^{mn} = 0. \quad (38)$$

Just as (34) yielded us the equation (35), from this identity we obtain the following equation for $\omega^{(ijk)l}$:

$$\omega^{(ijk)l} = a^l P^{(ijk)}. \quad (39)$$

This may be written as an equation for $\omega^{(ijk)}$:

$$a^l M^{(jk)l} - a^l P^{(ijk)} + 2\omega^{(ijk)} = 0. \quad (40)$$

We have here a situation which is more complicated than that above. Where as the identity (34) was sufficient to completely determine ω^{ijk} in terms of the multipole moments, this is not true of (38). The quantity $\omega^{(ijk)}$ is not completely determined. However to find the expression for the second term on the right-hand side of (32), the limited amount of information contained in equation (40) will be sufficient. In fact using the symmetry properties of the components of the affine connection, we have the following equalities:

$$\frac{1}{2} \Gamma_{jk,lm}^i \omega^{lmjk} = -\frac{1}{2} \Gamma_{lk,mj}^i \omega^{l(mjk)} = \frac{6}{4!} R_{klm}^i \omega^{l(mjk)}. \quad (41)$$

Therefore from (40) we find the following expression for the second term on the right-hand side of (32):

$$\frac{1}{2} \Gamma_{jk,lm}^i \omega^{lmjk} = \frac{1}{3!} R_{hjk,l}^i a^n M^{l(hk)} - \frac{2}{4!} R_{hkn,l}^i a^n P^{(jkl)}. \quad (42)$$

Placing (42) and (37) in (32) and then placing (32) and (30) in equation (16) of Section II yields the following differential equation for P^i correct to within terms containing multipole factors of order higher than the quadrupole:

$$\frac{dP^i}{dt} + \frac{1}{2!} R_{rjl}^i a^n M^{jk} + \frac{1}{3!} R_{rjl}^i a^n M^{l(jk)} + \frac{2}{3!} R_{hjk}^i \frac{dM^{hjk}}{dt} = 0. \quad (43)$$

This differential equation describes the evolution of P^i along the curve C .

We have performed all of the calculations in a particular coordinate system, a normal coordinate system with origin x where x is an arbitrary given point of the curve C . In this coordinate system, since the components of the affine connection vanish at x , covariant differentiation and ordinary differentiation are equal at this point. Therefore all of the terms occurring in equation (43) are components of tensors if we stipulate that d/dt be covariant differentiation along the curve C . Equation (43) is thus a tensorial equation.

To find the corresponding equation for M^{ij} we must make an additional assumption if we wish to obtain a differential equation which contains only the multipole moments of the matter distribution. We found above that we were unable to obtain the quantity ω^{ijkl} in terms of a^i and the quadrupole moment P^{jk} . The multipole moments involve the projection of the tensor T^{ij} onto the normal to the surface σ . That is, it is the vector $T^{ij}n_j$ which appears in the integrals in equations (1) and (2) defining the multipole moments. The quantity ω^{ijkl} is defined as an integral over σ containing the full tensor T^{ij} . The fact that $T^{ij}n_j$ does not in general determine T^{ij} is the reason why we cannot express ω^{ijkl} in terms of a^i and P^{jk} .

If we multiply both sides of equation (39) by n_k we find the following equation:

$$(\omega^{ijkl} - P^{ij}a^k)n_k = 0. \quad (44)$$

We recall that n_k is the unit normal to σ at the point x . What we shall assume is that the difference

$$\omega^{ijkl} - P^{ij}a^k, \quad (45)$$

which by equation (44) is normal to n_k , is of higher order than the quadrupole. Taub was led to make an equivalent assumption when he considered the motion of a point quadrupole (see [6], formula (3.3)).

To find the equation for M^{ij} we need to calculate $\Gamma_{kl,m}^{ij}\omega^{lmkl}$, which appears on the right-hand side of equation (33). We write this expression as the sum of three terms:

$$\begin{aligned} \Gamma_{kl,m}^{ij}\omega^{lmkl} = & \Gamma_{kl,m}^{ij}(\omega^{lmkl} - \omega^{kljm}) \\ & + \Gamma_{kl,m}^{ij}(\omega^{kljm} - P^{kl}a^m) \\ & + \Gamma_{kl,m}^{ij}P^{kl}a^m. \end{aligned} \quad (46)$$

The first term on the right-hand side of this equation is easily calculated. Using (39) we have

$$\Gamma_{kl,m}^j(\omega^{imk} - \omega^{klm}) = -\Gamma_{kl,m}^j\omega^{(ikl)m} = \frac{2}{3}R_{klm}^j a^{(i}p^{k)l)m}. \quad (47)$$

The second term is of higher order by assumption. Therefore we find the following expression for the term on the right-hand side of (33):

$$\Gamma_{kl,m}^{[i} \omega^{j]mkl} = \frac{2}{3!} R_{klm}^{[i} M^{j]ln(k} a^{l)} = \frac{2}{3!} M^{k[l} R_{klm}^{i]} a^m. \quad (48)$$

We have introduced the extra term on the right-hand side of equation (46) because of the simplification thus obtained in the final formula. We remark that if the ratio v/c is much less than one where v is a typical velocity of the sources of the field with respect to the body whose movement we are considering, then the introduction of the extra term makes the error we have made by assuming that the difference (45) is of higher order, correspondingly small. In fact, if we consider a normal coordinate system at x such that σ is given by $t = 0$, then what the assumption we have made means is that to within higher order multipoles, it is only the $(k, l) = (0, 0)$ components of ω^{ijkl} which do not vanish. Therefore the quantity

$$\Gamma_{kl,m}^j(\omega^{klm} - p^{kl}a^m) \approx \Gamma_{kl,0}^j(\omega^{kl0} - p^{kl})$$

is in general smaller by a factor v/c than the quantity

$$\Gamma_{kl,m}^j(\omega^{imkl} - p^{imk}a^l) \approx \Gamma_{00,a}^j(\omega^{ia00} - p^{ia0})$$

because of the extra time derivative.

Substituting (48) into (33) and then substituting (33) and (31) into equation (24) of Section II yields the following differential equation for M^{ij} correct up to quadrupole terms:

$$\frac{dM^{ij}}{dt} + a^{[i}p^{j]} + \frac{2}{3!}R_{klm}^{[i}M^{j]p(k}a^{l)} = 0. \quad (49)$$

This equation describes the evolution of M^{ij} along the curve C . As before with equation (43) it may be considered as tensorial with d/dt covariant differentiation along C .

We have two equations, describing the evolution of the energy-momentum vector and the angular momentum tensor along the curve C . In addition

we have a condition on C . The curve C was chosen such that the inner product of P^i and M^{ij} vanishes:

$$M^{ij}P_j = 0. \quad (50)$$

We have therefore in all 14 equations.

If we neglect the quadrupole moments we have exactly 14 unknowns: M^{ij} , P^i , a^i . In principle therefore we have a unique solution with given initial conditions [7]. In general if we cannot neglect the quadrupole terms then our system of equations is underdetermined.

If we multiply both sides of equation (49) by a_i we obtain the following relation between a^i and P^i :

$$P^i - P \cdot a a^i = \frac{dM^{ij}}{dt} a_j + \frac{2}{3!} R^i_{klm} M^{jlm} a^k a^l a^m - \frac{2}{3!} R_{jklm} M^{ink} a^l a^m. \quad (51)$$

This relation was remarked by Papapetrou [8] in the case of vanishing quadrupole moment.

Although formally we have included the case where the body contributes significantly to the field in which it is moving, the series expansion in terms of multipoles we have given would probably be of little use in this case. The multipole expansion in the case of a body of dimension d would be a good approximation only if the wave lengths λ of the field in which the body is moving are much longer than d . That is, if the inequality $d/\lambda \ll 1$. The exact equations (16) and (24) of course remain valid and eventually these equations could be used to consider self-acceleration effects.

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Renormalization and the Classical Electromagnetic One-Body Problem for an Extended Body.

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Summary. — A decomposition of the Maxwell stress-energy tensor into two terms is proposed corresponding to the decomposition of the electromagnetic force into radiative and nonradiative parts given by Dirac. This permits the nonradiative force to be expressed as the divergence of a nonradiative tensor. This tensor is added to the matter tensor to form the total or renormalized matter tensor.

1. — Introduction.

The process of renormalization was first introduced into classical physics by DIRAC⁽¹⁾ to deal with certain divergencies which appeared in the equations of motions of a charged point particle. DIRAC found that the force exerted on the particle by its own electromagnetic field could be decomposed into two terms. One was radiative in nature and remained bounded; the other was parallel to the acceleration vector and diverged. DIRAC added this second term to the time derivative of the energy-momentum vector of the particle and called the norm of the resulting vector the renormalized mass.

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(1) P. A. M. DIRAC: *Proc. Roy. Soc., A* **167**, 148 (1938).

We show here that this decomposition of the self-force can be carried out in a natural way for extended charged bodies (*). We also show that the term which tends to infinity in the limiting case of a point particle may be written as the divergence of a symmetric second-order tensor which is nonradiative in nature. We can therefore add this tensor to the matter tensor and consider the sum as describing the total stress-energy associated with the body. We find thereby a natural renormalization process for an extended body which includes the Dirac renormalization as a limiting case.

Since the self-force of a general extended body is a well-defined bounded quantity, we see that the process of renormalization which we consider here has nothing to do with the elimination of infinite self-energies. What it does is allow us to conveniently dispose of mathematically very complicated terms which do not play a direct role in the study of radiation.

We consider therefore an extended charged body in a flat space-time and we shall examine the forces exerted on it by its own electromagnetic field. We shall restrict the motion of the body by assuming that initially it is in a stationary state, that it executes an arbitrary motion during a finite interval of time and then that it returns to a stationary state, which is of course not in general identical to the initial stationary state. This restriction will play an important part in what follows. It allows us to decompose the space-time into well-defined regions where the fields have simple asymptotic behaviour. The study of this behaviour leads us then naturally to introduce a tensor which is important for the study of renormalization since it represents the part of the electromagnetic energy-momentum tensor which contains no radiation and which follows the body in its motion. One understands why other authors (2) working along lines very similar to ours did not distinguish it from the other quadratic tensors possible since they did not impose this restriction.

One may evidently find this hypothesis too restrictive and wish to weaken it. As a matter of fact it can be weakened by imposing only restrictions on the asymptotic behaviour of the acceleration, but it cannot be eliminated. This weakened hypothesis corresponds to what other authors (3) have been forced to impose to deal with difficulties which have essentially the same origin.

(*) We are not thinking here necessarily of a solid body but rather of a material distribution of finite extension possessing any given number of degrees of freedom. In the present study it is not necessary to specify them, but one must of course do so when one wishes to discuss a concrete problem.

(2) See for example H. J. BIRABIA and HARISH-CHANDRA: *Proc. Roy. Soc., A* 185, 250 (1944).

(3) See for example F. ROUILLICH: *Classical Charged Particles*, Sect. 6.4 (New York, 1965) for a discussion of the problem of self-acceleration in the Lorentz-Dirac model of the electron.

The paper consists of three parts. Section 2 describes the asymptotic behaviour of the Maxwell stress-energy tensor. Section 3 is the main part. It gives the definition of a second-order tensor and uses it to decompose the Maxwell stress-energy tensor into two terms. Section 4 gives the definition of the renormalized energy-momentum vector and angular-momentum tensor and discusses their equations of motion. A discussion of the important particular limiting case of a rotating bar whose cross-section tends to zero will be given in a subsequent paper.

2. - Asymptotic behaviour of the Maxwell stress-energy tensor.

Figure 1 shows the world tube of the body with two light cones chosen in such a way that the region of accelerated motion of the body is contained between them.

Let φ_-^α and φ_+^α be respectively the retarded and advanced solutions of the Maxwell equation

$$\square \varphi^\alpha = 4\pi J^\alpha,$$

which vanish at infinity. They are given by

$$\varphi_\pm^\alpha(x^i, x^0) = \int \frac{1}{R} J^\alpha(y^i, x^0 \pm R) d^3y.$$

J^α is the source vector and $R^2 = (y^i - x^i)^2$.

Since we are considering the one-body problem

we shall have $J^\alpha \neq 0$ only inside the world tube of the body. Let $F_\pm^{\alpha\beta} = \varphi_\pm^{\beta,\alpha} - \varphi_\pm^{\alpha,\beta}$ be the corresponding retarded and advanced Maxwell fields.

One deduces immediately the following asymptotic behaviour for $F^{\alpha\beta}$: the retarded field $F_-^{\alpha\beta}$ vanishes as $1/R$ in the region B of Fig. 1 and as $1/R^2$ elsewhere; the advanced field $F_+^{\alpha\beta}$ vanishes as $1/R$ in the region D and as $1/R^2$ elsewhere. Without our hypothesis that the accelerated motion lasts only a finite length of time it would not be possible to decompose the space-time into these regions where the field vanishes at infinity alternatingly as $1/R$ or $1/R^2$.

The physically significant Maxwell stress-energy tensor is constructed with the help of the retarded field:

$$(1) \quad -4\pi\tau^{\alpha\beta} = F_-^{\alpha\gamma} F_-^{\beta}_{\gamma} - \frac{1}{4} g^{\alpha\beta} F_-^{\gamma\delta} F_-^{\delta}_{\gamma}.$$

From the asymptotic behaviour of $F_-^{\alpha\beta}$, one sees immediately that $\tau^{\alpha\beta}$ vanishes as $1/R^2$ in the region B and as $1/R^4$ elsewhere.

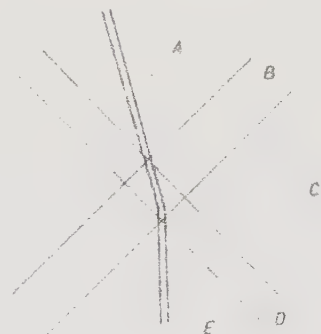


Fig. 1.

3. - Decomposition of the Maxwell tensor into radiative and nonradiative parts. Renormalization.

We shall now decompose the Maxwell stress-energy tensor $\tau^{\alpha\beta}$ into two physically different parts. To this end we define the tensor $\hat{\tau}^{\alpha\beta}$ by

$$(2) \quad -4\pi\hat{\tau}^{\alpha\beta} = \frac{1}{2}(F_{-}^{\alpha\gamma}F_{+\gamma}^{\beta} + F_{+}^{\alpha\gamma}F_{-\gamma}^{\beta}) - \frac{1}{2}g^{\alpha\beta}F_{+}^{\gamma\delta}F_{-\gamma\delta}.$$

One sees that $\hat{\tau}^{\alpha\beta}$ vanishes as $1/R^3$ in the regions B and D and as $1/R^4$ elsewhere and thus it is nonradiative everywhere. Note that without our restriction on the motion it would not have this property. Moreover one easily sees that $\hat{\tau}^{\alpha\beta}$ is the only tensor which has this property and reduces to the Maxwell tensor for a stationary field.

It is known that

$$(3) \quad \tau^{\alpha\beta}_{;\beta} = -F_{-}^{\alpha\beta}J_{\beta},$$

where the right-hand side is the density of force exerted on the body by its own electromagnetic field. A calculation completely analogous to that used to derive the above formula yields the following expression for the divergence of $\hat{\tau}^{\alpha\beta}$:

$$(4) \quad \hat{\tau}^{\alpha\beta}_{;\beta} = -\frac{1}{2}(F_{-}^{\alpha\beta} + F_{+}^{\alpha\beta})J_{\beta}.$$

Subtracting (4) from (3) gives

$$(5) \quad (\tau^{\alpha\beta} - \hat{\tau}^{\alpha\beta})_{;\beta} = -\frac{1}{2}(F_{-}^{\alpha\beta} - F_{+}^{\alpha\beta})J_{\beta}.$$

We are now led to decompose the force $F_{-}^{\alpha\beta}J_{\beta}$ in two terms

$$(6) \quad F_{-}^{\alpha\beta}J_{\beta} = \frac{1}{2}(F_{-}^{\alpha\beta} + F_{+}^{\alpha\beta})J_{\beta} + \frac{1}{2}(F_{-}^{\alpha\beta} - F_{+}^{\alpha\beta})J_{\beta}.$$

DIRAC has shown that the first term is the force which is physically significant when one is considering the radiation emitted by a point particle. Only this term must therefore appear in the equations of motion. The second term will play a different role. Formula (4) shows that this second term is the divergence of $\hat{\tau}^{\alpha\beta}$. It suffices therefore to decompose $\hat{\tau}^{\alpha\beta}$ as follows:

$$(7) \quad \hat{\tau}^{\alpha\beta} = (\tau^{\alpha\beta} - \hat{\tau}^{\alpha\beta}) + \hat{\tau}^{\alpha\beta}.$$

The first term is the radiative one: it vanishes at infinity as $1/R^2$ in the

region B ; and its divergence is the radiative force according to eq. (5). We shall therefore consider the second term $\hat{\tau}^{\alpha\beta}$ as describing the electromagnetic energy-momentum which stays associated with the body and is not carried away by radiation.

The conservation of the total energy and momentum is expressed by equating the divergence of the sum of the matter tensor $T^{\alpha\beta}$ and the Maxwell stress-energy tensor $\tau^{\alpha\beta}$ to zero:

$$(8) \quad (T^{\alpha\beta} + \tau^{\alpha\beta})_{;\beta} = 0.$$

To the extent that we are interested in studying the motion of the body as a whole we see from the preceding discussion that we are justified in adding $\hat{\tau}^{\alpha\beta}$ to the matter tensor $T^{\alpha\beta}$ and writing the conservation law as

$$(9) \quad (T^{\alpha\beta} + \hat{\tau}^{\alpha\beta})_{;\beta} = -(\tau^{\alpha\beta} - \hat{\tau}^{\alpha\beta})_{;\beta}.$$

4. — Equations of motion.

We shall use the tensor $T^{\alpha\beta} + \hat{\tau}^{\alpha\beta}$ to form the vector $P^\alpha(\sigma)$ by integration over a spacelike hyperplane σ :

$$(10) \quad P^\alpha(\sigma) = \int_\sigma (T^{\alpha\beta} + \hat{\tau}^{\alpha\beta}) n_\beta d\sigma.$$

We shall call this vector the renormalized or total energy-momentum vector of the body. The fact that the body has been assumed stationary outside of a finite interval of time assures us that outside of a bounded region of σ , $\hat{\tau}^{\alpha\beta}$ vanishes as $1/R^4$ and therefore that the quantity

$$(11) \quad \delta P^\alpha = \int_\sigma \hat{\tau}^{\alpha\beta} n_\beta d\sigma$$

remains bounded. Therefore our expression for $P^\alpha(\sigma)$ is well defined. It is to be noted that in general P^α and δP^α may not be parallel. However in the limiting case of a Dirac particle they are both parallel to the velocity vector and our renormalization reduces to a change in the length of P^α that is to a mass renormalization.

In a similar manner we define the renormalized angular momentum with

respect to a point x_0 as

$$(12) \quad M^{\alpha\beta}(\sigma, x_0) = \int_{\sigma} X^{\alpha} (T^{\beta\gamma} + \hat{\tau}^{\beta\gamma}) n_{\gamma} d\sigma,$$

where X^{α} is the vector joining the point of integration to x_0 . We note however that now the correction term

$$(13) \quad \delta M^{\alpha\beta}(\sigma, x_0) = \int_{\sigma} X^{\alpha} \hat{\tau}^{\beta\gamma} n_{\gamma} d\sigma$$

does not converge absolutely. However it is expected that in important special cases it will converge because of symmetric properties of the leading term in the asymptotic expansion of the tensor $\hat{\tau}^{\alpha\beta}$.

Using eq. (5) we can rewrite eq. (9) in the form

$$(14) \quad (T^{\alpha\beta} + \hat{\tau}^{\alpha\beta})_{;\beta} = \frac{1}{2} (P_{-}^{\alpha\beta} - P_{+}^{\alpha\beta}) J_{\beta}.$$

This equation may be used to derive differential equations satisfied by the renormalized quantities P^{α} and $M^{\alpha\beta}$ defined above.

To proceed further in the derivation of the equations of motion a choice must be made for the family of hypersurfaces σ used to define P^{α} and $M^{\alpha\beta}$. Also a center-of-mass line in the interior of the world tube of the body must be chosen to represent the motion. For a further discussion in the case of an extended body see for example (4). For a discussion of the limiting case of a point particle see for example ref. (2).

Finally we remark that if we integrate eq. (14) over the entire space-time, we obtain the following expression for the total energy-momentum radiated by the body:

$$(15) \quad \Delta P^{\alpha} = \frac{1}{2} \int (P_{-}^{\alpha\beta} - P_{+}^{\alpha\beta}) J_{\beta} d^4x.$$

One sees immediately that the integrand is different from zero only in the part of the world tube between the two cones in Fig. 1. An analogous result was found previously for the case of the linearized gravitational field by one of the authors (4).

In a subsequent paper one of the authors (J.M.) shall treat in detail the particular limiting case of a rotating bar whose cross-section tends to zero.

(4) J. MADORE: *Compt. Rend. Acad. Sci.*, 283, 746 (1966).

(5) A. PAPAPETROU: *Ann. Inst. Henri Poincaré*, 1, 117 (1964).

Renormalization of the Moment of Inertia for the Classical Electromagnetic One-body Problem

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Dedicated to Professor PASCUAL JORDAN on the occasion of his 65th birthday

A general process of renormalization of the energy-momentum vector and the angular-momentum tensor is given. It is shown that in the simple limiting case of a charged line segment rotating about a fixed point of symmetry, this process reduces to a renormalization of the moment of inertia analogous to the mass renormalization given by DIRAC for the limiting case of a point particle.

In a preceding paper¹ a general process of renormalization of the energy-momentum vector and the angular-momentum tensor is given. We show in this article that in the simple limiting case of a charged line segment rotating about a fixed point of symmetry, this process reduces to a renormalization of the moment of inertia analogous to the mass renormalization given by DIRAC for the limiting case of a point particle². For completeness we include a derivation of DIRAC's result from the point of view of¹.

In section I we briefly review the results of¹. Since we are dealing with a conserved matter tensor, the choice of hyperplane over which we integrate to form the energy-momentum vector and angular-momentum tensor is important. In section II a canonical method of choosing a hyperplane at each point is given. Also a timelike curve to represent the body movement is defined. In section III we give some preliminary formulae and in sections IV and V we carry out the renormalization calculations in the two cases to be considered.

I.

We consider an isolated extended charged body in a flat space-time and assume that it undergoes accelerated motion for only a finite interval of time. (See¹ for a discussion of this assumption.) Let ϱ be the charge density of the body. We assume the current to be of the form ϱu^i .

Let $\varphi_{\pm}^i$ be the advanced and retarded solutions of the wave equation

$$\square \varphi^i = 4\pi \varrho u^i$$

which vanish at infinity. From these we form the advanced and retarded MAXWELL fields:

$$F_{\pm}^{ij} = \varphi_{\pm}^{j,i} - \varphi_{\pm}^{i,j}.$$

The MAXWELL stress-energy tensor is defined in terms of the retarded field:

$$-4\pi \tau^{ij} = F_{-}^{ik} F_{-k}^j - \frac{1}{2} g^{ij} F_{-}^{kl} F_{-kl}. \quad (1)$$

Let T^{ij} be the matter tensor of the body. Using the identity

$$\tau_{ij,j} = -F_{-}^{ij} \varrho u_j, \quad (2)$$

the conservation laws

$$(T^{ij} + \tau^{ij})_{,j} = 0 \quad (3)$$

may be written in the form

$$T_{ij,j} = F_{-}^{ij} \varrho u_j. \quad (4)$$

Using the advanced as well as the retarded MAXWELL field, this equation may be rewritten as

$$T_{ij,j} = \frac{1}{2} (F_{-}^{ij} + F_{+}^{ij}) \varrho u_j + \frac{1}{2} (F_{-}^{ij} - F_{+}^{ij}) \varrho u_j. \quad (5)$$

In¹ the tensor τ^{ij} , defined by

$$-4\pi \tau^{ij} = \frac{1}{2} (F_{-}^{ik} F_{-k}^j + F_{+}^{ik} F_{-k}^j) - \frac{1}{2} g^{ij} F_{-}^{kl} F_{+kl} \quad (6)$$

is introduced and it is shown that the divergence of τ^{ij} is precisely the first term on the right hand side of Eq. (5):

$$-\tau_{ij,j} = \frac{1}{2} (F_{-}^{ij} + F_{+}^{ij}) \varrho u_j. \quad (7)$$

DIRAC identified the second term on the right hand side of Eq. (5) as being the term responsible for radiation effects in the limiting case of a point particle and in¹ it is shown that for an arbitrary body,

¹ E. ARNOUS, J. MADORE, and A. PAPAETROU, Nuovo Cimento [1967] (To appear).

² P. A. M. DIRAC, Proc. Roy. Soc. London A 167, 148 [1938].

the tensor τ^{ij} is nonradiative provided the duration of accelerated motion is finite. Therefore to the extent that we are not interested in studying the details of the electromagnetic forces in the interior and in the neighborhood of the body but rather those forces which are directly responsible for the movement of the body as a whole and for radiation, we are justified in using the identity (7) to write Eq. (5) in the form

$$(T^{ij} + \tau^{ij})_{,j} = \frac{1}{2} (F_{-}^{ij} - F_{+}^{ij}) \otimes u_j, \quad (8)$$

and in considering $T^{ij} + \tau^{ij}$ as the total or renormalized matter tensor of the body.

Let x_0 be a point in the interior of the body and σ an arbitrary spacelike hyperplane containing x_0 . We form the renormalized energy-momentum vector and angular-momentum tensor the usual way but using $T^{ij} + \tau^{ij}$ instead of T^{ij} :

$$P^i(\sigma, x_0) = \int_{\sigma} (T^{ij} + \tau^{ij}) n_j d\sigma, \quad (9)$$

$$M^{ij}(\sigma, x_0) = \int_{\sigma} X^i (T^{jk} + \tau^{jk}) n_k d\sigma, \quad (10)$$

where X^i is the vector joining the point of integration to x_0 and n^i the unit normal to σ .

These equations may be written in the form

$$P^i(\sigma, x_0) = P_{(0)}^i(\sigma, x_0) + \delta P^i(\sigma, x_0), \quad (11)$$

$$M^{ij}(\sigma, x_0) = M_{(0)}^{ij}(\sigma, x_0) + \delta M^{ij}(\sigma, x_0), \quad (12)$$

where $P_{(0)}^i$ and $M_{(0)}^{ij}$ are the usual energy-momentum vector and angular-momentum tensor. We are interested here in finding the form of the correction terms δP^i and δM^{ij} in two particular simple limiting cases.

II.

For each point x_0 in the world tube of the body we have introduced a family of vectors $P^i(\sigma, x_0)$ depending on the choice of hyperplane σ . We shall now choose from this family, one vector which we shall take to be the energy-momentum vector at the point x_0 . (See for example ³ for a more detailed discussion in the case of General Relativity and ⁴ for a discussion of the case of a point particle.)

Let n^i be a timelike vector and σ_n be the hyperplane whose unit normal is parallel to n^i . Since we are considering macroscopic bodies, physical considerations require that $P^i(\sigma_n, x_0)$ be timelike. There-

fore the map

$$n^i \rightarrow P^i(\sigma_n, x_0) / |P(\sigma_n, x_0)|$$

is a map of the forward light cone into itself. It can be shown using BROUWER's Fixed Point theorem that this map has a fixed point. We shall assume that it is unique and designate simply as $P^i(x_0)$ the energy-momentum vector so formed. By construction therefore $P^i = |P|n^i$. We designate as $M^{ij}(x_0)$ the angular-momentum vector formed using this hyperplane.

The two cases we are going to consider are the point particle, which we shall take as the limit of a body whose diameter tends to zero and the line segment rotating about a fixed point, which we shall take as the limit of a bar whose cross-sectional width tends to zero. Since these two cases involve extended bodies it is convenient to have a timelike curve C in the interior of the world tube to represent the motion. We construct this curve C as follows.

It is easily seen that the vector field $M^{ij}P_j$ which is defined everywhere in the world tube of the body, is a spacelike field. Using the Fixed Point Theorem previously mentioned and the fact that the matter density is nowhere negative, one shows that it has a zero on every spacelike hyperplane. One can also show that this zero is unique. We designate by C the locus of the zeros. One sees immediately that C is a differentiable timelike curve which generalizes the center-of-mass line of Newtonian Physics. Let t be the geodesic parameter along C and a^i the unit tangent vector. We shall assume for simplicity that a^i and P^i are parallel.

III.

Let x and x' be two arbitrary points of a spacelike hyperplane σ in the interior of the world tube of the body. Let $x_{\pm}$ be the points respectively on the future and past light cones with vertex x' which lie on the world line representing the motion of the matter through x .

Let u^i be the particle velocity vector at x and define the vectors $u_{\pm}^i$ by

$$u_{\pm}^i(x, x') = u^i(x_{\pm}).$$

$$\text{Set } r^i = x'^i - x^i \text{ and } r_{\pm}^i = x'^i - x_{\pm}^i.$$

³ J. MADORE, C. R. Acad. Sci. Paris **263**, 746 [1966].

⁴ F. ROHLICH, Classical Charged Particles, Addison-Wesley, London 1965.



The advanced and retarded solutions of the wave equation, $\varphi_{\pm}^i$ may be written in the form

$$\varphi_{\pm}^i = \mp \int_{\sigma} \frac{\partial u_{\pm}^i}{\partial r_{\pm}} u_{\pm} n d\sigma, \quad (13)$$

A straightforward calculation shows that the advanced and retarded MAXWELL fields are given by

$$F_{\pm}^{ij} = \mp \int_{\sigma} \frac{\partial u_{\pm}^i}{\partial r_{\pm}} \frac{\partial u_{\pm}^j}{\partial r_{\pm}} \left(\dot{u}_{\pm}^i r_{\pm}^j + \frac{1 - r_{\pm}^i \cdot \dot{u}_{\pm}^i}{r_{\pm}^i \cdot u_{\pm}^i} u_{\pm}^j r_{\pm}^i \right) d\sigma, \quad (14)$$

where $\dot{u}^i = u_{,j}^i u^j$ is the acceleration vector. These may be written in the form

$$F_{\pm}^{ij} = \mp \int_{\sigma} \omega_{\pm}^{ij} u_{\pm} n d\sigma, \quad (15)$$

where $\omega_{\pm}^{ij}$ are defined by

$$\omega_{\pm}^{ij} = \frac{1}{r_{\pm}^i \cdot u_{\pm}^i} \left(\dot{u}_{\pm}^i r_{\pm}^j + \frac{1 - r_{\pm}^i \cdot \dot{u}_{\pm}^i}{r_{\pm}^i \cdot u_{\pm}^i} u_{\pm}^j r_{\pm}^i \right). \quad (16)$$

We wish to calculate the correction terms

$$\delta P^i = \int_{\sigma} \tau^{ij} n_j d\sigma, \quad (17)$$

and

$$\delta M^{ij} = \int_{\sigma} X^{(i} \tau^{j)k} n_k d\sigma, \quad (18)$$

along the curve C. It is more convenient to calculate the right hand sides of (17) and (18) if we first differentiate both sides with respect to t and use Eq. (7) to eliminate τ^{ij} . In the case when C is not a straight line, care must be taken in performing this derivation since the orientation of the hyperplane σ will depend on t .

By definition

$$\frac{d(\delta P^i)}{dt} = \lim_{t' \rightarrow t} \frac{\delta P^i(t') - \delta P^i(t)}{t' - t}.$$

This gives, using GAUSS' Theorem on the right hand side of (17),

$$\frac{d(\delta P^i)}{dt} = \lim_{t' \rightarrow t} \int_V \tau^{ij}_{,j} dv, \quad (19)$$

where V is the volume enclosed by the two hyperplanes $\sigma(t')$ and $\sigma(t)$. We have here used the fact that τ^{ij} vanishes as $1/R^3$ at infinity along spacelike hyperplanes. A straightforward calculation shows that

$$dv = (t' - t) (1 - X \cdot \dot{a}) d\sigma + o((t' - t)^2) \quad (20)$$

where $\dot{a}^i = a^i_{,j} u^j$. Therefore we have

$$\frac{d(\delta P^i)}{dt} = \int_{\sigma} \tau^{ij}_{,j} (1 - X \cdot \dot{a}) d\sigma. \quad (21)$$

Using (7) this gives

$$\frac{d(\delta P^i)}{dt} = - \frac{1}{2} \int_{\sigma} \partial (F_{-}^{ij} + F_{+}^{ij}) u_j (1 - X \cdot \dot{a}) d\sigma. \quad (22)$$

Similarly for the correction term δM^{ij} we find the following derivative along C:

$$\begin{aligned} \frac{d(\delta M^{ij})}{dt} &= - \frac{1}{2} \int_{\sigma} \partial X^{(i} (F_{-}^{j)k} + F_{+}^{j)k}) u_k (1 - X \cdot \dot{a}) d\sigma. \end{aligned} \quad (23)$$

We have here used the assumption of section II that P^i and a^i are parallel.

Using (15) and defining φ^{ij} by

$$\varphi^{ij} = \frac{1}{2} (\omega_{-}^{ij} - \omega_{+}^{ij}), \quad (24)$$

we obtain the following expression for the derivatives of the two correction terms along C:

$$\frac{d(\delta P^i)}{dt} = - \iint \partial \varphi^{ij} u_j (1 - X' \cdot \dot{a}) u \cdot a d\sigma d\sigma', \quad (25)$$

$$\frac{d(\delta M^{ij})}{dt} = - \iint \partial X^{(i} \varphi^{j)k} u_k (1 - X' \cdot \dot{a}) u \cdot a d\sigma d\sigma'. \quad (26)$$

The functions φ^{ij} are expressed in terms of the vectors $u_{\pm}^i, \dot{u}_{\pm}^i, r_{\pm}^i$ defined at $x_{\pm}$ on the future and past light cones with vertex x' . We wish to express them in terms of $u^i, \dot{u}^i$ and r^i , vectors defined at x on the hyperplane σ . To do this we must expand the advanced and retarded quantities in a TAYLOR series about the point x . This calculation is straightforward but tedious and the details are given in the appendix.

The result is:

$$\begin{aligned} \varphi^{ij} &= \frac{1 - r \cdot \dot{u}}{R^3} \left(1 - \frac{1}{2} R^2 \dot{a}^2 \right) r^{(i} u^{j)} + \frac{u^{(i} \dot{u}^{j)} - r^{(i} \dot{u}^{j)}}{2 R (1 - r \cdot \dot{u})} \\ &\quad + (r \cdot u / R^3) (r^{(i} \dot{u}^{j)} - \frac{1}{2} r \cdot u u^{(i} \dot{u}^{j)}) \\ &\quad + (1/R) o(r \cdot u) + o(|r|), \end{aligned} \quad (27)$$

where R is given by $R = \sqrt{r \cdot u^2 - (1 - r \cdot \dot{u})^2 r^2}$ and by $o(u \cdot r)$ we designate a function which vanishes with $u \cdot r$.

IV.

We are now in a position to carry out the calculation of the correction term δP^i and δM^{ij} in the two limiting cases to be considered.

First we consider the case of a point particle. We calculate only the correction term δP^i in this case. We consider the point particle as the limit of an arbitrary body of diameter d with a charge distribution symmetric about the center of mass point x_0 and which we assume to be in translational movement. That is we assume that the velocity vector field u^i is constant on σ and equal to a^i .

Therefore since r^i lies in σ and a^i is normal to σ we have $u \cdot r = 0$ and it is easily seen from formula (27) that the quantity $\varphi^{ij} u'_j u \cdot a$ appearing in the integrand of Eq. (25) is given by

$$\varphi^{ij} u'_j u \cdot a = (1 - r \cdot \dot{a}) (R^{-3} + \frac{1}{2} \dot{a}^2 / R) r^i - \dot{a}^i / [2 R (1 - r \cdot \dot{a})] + o(|r|). \quad (28)$$

Since $r \cdot u = 0$ we have $R = |r| \sqrt{1 - r \cdot \dot{a}}$. The vector $X^i = x^i - x_0^i$ may be written as

$$X^i = \frac{1}{2} r^i + t^i$$

where $t^i = \frac{1}{2} (x'^i + x^i) - x_0^i$ is a symmetric function of the two variables of integration. Therefore one deduces immediately the following expression for the integrand of Eq. (25):

$$\varrho \varphi^{ij} u'_j (1 - X' \cdot \dot{a}) u \cdot a = \varrho \varphi' \left(-\frac{\dot{a}^i}{2|r|} + \frac{3 \dot{a}^2 r^i}{8|r|} - \frac{r \cdot \dot{a} a^i}{2|r|} + \frac{t \cdot \dot{a}}{2|r|} \left(\dot{a}^i - \frac{r \cdot \dot{a} r^i}{|r|^2} \right) \right) + o(|r|). \quad (29)$$

Since r^i is antisymmetric in the two variables of integration the second and third terms on the right hand side contribute nothing to the integration. From the assumed symmetry of the charge density about the point x_0 one sees immediately that the fourth term contributes also nothing. Therefore the expression $(2\alpha) = d(\delta P^i)/dt$ becomes

$$\frac{d(\delta P^i)}{dt} = \frac{1}{2} \left(\iint \frac{\varrho \varphi'}{|r|} d\sigma d\sigma' \right) \dot{a}^i + o(d), \quad (30)$$

where $o(d)$ is a quantity which vanishes with d .

The coefficient of $\dot{a}^i$ is the electrostatic self-energy of the body, $E_{(e)}$. We have therefore

$$\frac{d(\delta P^i)}{dt} = E_{(e)} \dot{a}^i + o(d). \quad (31)$$

Since P^i and a^i have been assumed to be parallel, Eq. (11) may in this case be written in the form

$$m a^i = (m_{(0)} + E_{(e)}) a^i + o(d), \quad (32)$$

where $m_{(0)} = |P_{(0)}|$ and $m = m_{(0)} + E_{(e)}$ is the renormalized mass given by DIRAC.

V.

The second case to be considered is that of a charged line segment rotating about a fixed point of symmetry. Here we calculate only the correction term δM^{ij} . In fact we calculate only the space-like components $\delta M^{3\beta}$ of this tensor; the components $\delta M^{0\alpha}$ vanish by symmetry. The line C will be in

this case the world-line of the point of symmetry. We choose for convenience a system of coordinates such that it is the line $x^\alpha = 0$ ($\alpha = 1, 2, 3$).

We consider the line segment as the limit of a bar with a point symmetry whose cross-sectional width d tends to zero. The situation is more complicated in this case and we shall only be able to calculate $\delta M^{3\beta}$ to within terms which remain bounded as d tends to zero and then only under rather restrictive assumptions on the asymptotic behaviour of the charge distribution.

We see from formula (27) that the quantity $\varphi^{jk} u'_k$ appearing in the integrand of Eq. (26) is given by

$$\varphi^{jk} u'_k = ((1 - r \cdot \dot{u})/R^3) (r^j u \cdot u' - u^j r \cdot u') - \ddot{u}^j r \cdot u^2 / 2 R^3 - \ddot{u}^j / 2 R + o(1), \quad (33)$$

where $o(1)$ is a quantity which remains bounded in the limit as $|r| \rightarrow 0$. We have used the fact that $u' \cdot \ddot{u} = o(|r|)$ and that $u' \cdot u = 1 + o(|r|)$.

In the limit $d = 0$ the quantity $r \cdot u$ is zero since r^i is a vector in the line segment and u^i is perpendicular to it. However, since this quantity appears as a factor in integrals which diverge in general as $d \rightarrow 0$, the asymptotic behaviour of the integral of the terms involving $r \cdot u$ on the right hand side of Eq. (33) is obscure. It will in fact depend heavily on the asymptotic behaviour of the charge density. We shall not enter into a discussion of this point but shall assume the asymptotic behaviour of ϱ which is most con-

venient. One sees immediately that, since we have the equality $r \cdot u = 0$ when both variables of integration lie on the plane defined by the axis of rotation and the axis of the rod, if we assume the charge to become concentrated in the limit on this plane we may set $r \cdot u = 0$ under the integration sign.

Therefore neglecting relativistic corrections and taking the three dimensional dual we obtain for Eq. (26) the following

$$\begin{aligned} d(\delta M)/dt = & \frac{1}{2} \iint (\rho \rho' / |r|) \mathbf{X} \times \dot{\mathbf{u}} \, d\sigma \, d\sigma' \\ & - \frac{1}{2} \iint (\rho \rho' / |r|^3) \mathbf{X}' \times \mathbf{X} \, r \cdot \dot{\mathbf{u}} \, d\sigma \, d\sigma' \quad (35) \\ & + \iint (\rho \rho' / |r|^3) \mathbf{X}' \times \mathbf{X} \, d\sigma \, d\sigma' + o(1). \end{aligned}$$

By $o(1)$ we designate here a quantity which remains bounded in the limit $d \rightarrow 0$.

Since $\mathbf{X}' \times \mathbf{X}$ is antisymmetric in the two variables of integration one sees that the last term on the right hand side vanishes.

Also from our assumptions on the asymptotic behaviour of the charge distribution the second integral can give only a contribution perpendicular to the rotation vector and therefore must vanish because of the point symmetry of the bar. We are therefore left with the first term. Since the rotation vector $\boldsymbol{\omega}$ and its derivative $\dot{\boldsymbol{\omega}}$ are parallel, we have $\mathbf{X} \times \dot{\mathbf{u}} = \mathbf{X}^2 \boldsymbol{\omega}$. Therefore eq. (35) becomes

$$d(\delta M)/dt = \frac{1}{2} (\iint (\rho \rho' / |r|) \mathbf{X}^2 \, d\sigma \, d\sigma') \dot{\boldsymbol{\omega}} + o(1). \quad (36)$$

The coefficient of $\dot{\boldsymbol{\omega}}$ is the moment of inertia $I_{(0)}$ of the electrostatic energy about the origin. We have therefore

$$d(\delta M)/dt = I_{(0)} \dot{\boldsymbol{\omega}} + o(1). \quad (37)$$

If $I_{(0)}$ is constant along C , (37) may be integrated to give the following expression for the dual of the space-like components of the correction term δM^i :

$$\delta M = I_{(0)} \boldsymbol{\omega} + o(1). \quad (38)$$

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Appendix

We give here a derivation of formula (27). Let $s_{\pm}$ be the proper time along the particle line between x and $x_{\pm}$.

We have the following expansions in terms of the proper time interval $s_{\pm}$:

$$r_{\pm}^i = r^i - s_{\pm} u^i - \frac{1}{2} s_{\pm}^2 \ddot{u}^i - \frac{1}{6} s_{\pm}^3 \dddot{u}^i + o(s_{\pm}^4), \quad (1A)$$

$$u_{\pm}^i = u^i + s_{\pm} \dot{u}^i + \frac{1}{2} s_{\pm}^2 \ddot{u}^i + o(s_{\pm}^3), \quad (2A)$$

$$\ddot{u}_{\pm}^i = \ddot{u}^i + s_{\pm} \dddot{u}^i + o(s_{\pm}^2). \quad (3A)$$

Using these we obtain the following expansions for $r_{\pm}^i u_{\pm}^j$ and for $r_{\pm}^i u_{\pm}^j$:

$$r_{\pm}^i u_{\pm}^j = r^i u^j - s_{\pm} u^i u^j + s_{\pm} r^i \ddot{u}^j - s_{\pm}^2 u^i \ddot{u}^j + o(s_{\pm}^3), \quad (4A)$$

$$r_{\pm}^i u_{\pm}^j = r^i u^j + s_{\pm} r^i \dot{u}^j - \frac{1}{2} s_{\pm}^2 u^i \ddot{u}^j + \frac{1}{2} s_{\pm}^2 r^i \ddot{u}^j - \frac{1}{6} s_{\pm}^3 u^i \dddot{u}^j + o(s_{\pm}^4). \quad (5A)$$

Equations (4A) and (5A) may be used in turn to give a partial expansion of the right hand side of Eq. (16) which defines the quantities $\omega_{\pm}^{ij}$:

$$\begin{aligned} \omega_{\pm}^{ij} = & A_{\pm} r^i u^j + \left(\frac{1}{r_{\pm} \cdot u_{\pm}^2} + s_{\pm} A_{\pm} \right) r^i \dot{u}^j - \left(\frac{s_{\pm}}{r_{\pm} \cdot u_{\pm}^2} + \frac{s_{\pm}^2 A_{\pm}}{2} \right) u^i \ddot{u}^j \\ & + \left(\frac{s_{\pm}}{r_{\pm} \cdot u_{\pm}^2} + \frac{s_{\pm}^2 A_{\pm}}{2} \right) r^i \ddot{u}^j - \left(\frac{s_{\pm}^2}{r_{\pm} \cdot u_{\pm}^2} + \frac{s_{\pm}^3 A_{\pm}}{3} \right) u^i \ddot{u}^j + o(s_{\pm}). \end{aligned} \quad (6A)$$

We have set

$$(1 - r_{\pm} \cdot \ddot{u}_{\pm}) / r_{\pm} \cdot u_{\pm}^3 = A_{\pm}.$$

To calculate the coefficients in this expansion, we need the following two expressions which are obtained from (1A), (2A) and (3A):

$$1 - r_{\pm} \cdot \ddot{u}_{\pm} = 1 - r \cdot \ddot{u} - s_{\pm} r \cdot \dddot{u} - \frac{1}{2} s_{\pm}^2 \ddot{u}^2 + o(s_{\pm}^3), \quad (7A)$$

$$r_{\pm} \cdot u_{\pm} = r \cdot u - s_{\pm} (1 - r \cdot \ddot{u}) + \frac{1}{2} s_{\pm}^2 r \cdot \ddot{u} + \frac{1}{6} s_{\pm}^3 \ddot{u}^2 + o(s_{\pm}^4). \quad (8A)$$

The proper time intervals $s_{\pm}$ are obtained as functions of r^i by using the fact that the vectors $r_{\pm}^i$ are null. That is that

$$(r^i - s u^i - \frac{1}{2} s^2 \ddot{u}^i - \frac{1}{6} s^3 \dddot{u}^i)^2 = o(s^4). \quad (9A)$$

The solutions to this equation are

$$s_{\pm} = ((r \cdot u \pm R)/(1 - r \cdot \dot{u})) (1 \pm \frac{1}{2} R r \cdot \ddot{u} + \frac{1}{24} R^2 \ddot{u}^2) + R^2 o(r \cdot u) + o(|r|^4), \quad (10 A)$$

where we have set $R = \sqrt{r \cdot \dot{u}^2 - (1 - r \cdot \dot{u})^2}$ and where by $o(u \cdot r)$ we designate a function which vanishes with $u \cdot r$.

Putting this expression for $s_{\pm}$ in (8 A) we obtain the following expansion for $r_{\pm} \cdot u_{\pm}$:

$$r_{\pm} \cdot u_{\pm} = \mp R + \frac{1}{2} R^2 r \cdot \ddot{u} \pm \frac{1}{6} R^3 \ddot{u}^2 + o(|r|^4) + R^2 o(r \cdot u). \quad (11 A)$$

This and a similar expansion for $1 - r_{\pm} \cdot \dot{u}_{\pm}$ give the following expansion for $A_{\pm}$:

$$A_{\pm} = ((1 - r \cdot \dot{u})/(\mp R^3)) (1 - \frac{1}{2} R^2 \ddot{u}^2) + o(|r|^0) + (1/R) o(r \cdot u). \quad (12 A)$$

The expansions of the retarded and advanced coefficients appearing in (6 A) are, using (10 A), (11 A) and (12 A):

$$1/r_{\pm} \cdot u_{\pm}^2 + s_{\pm} A_{\pm} = \mp r \cdot u/R^3 + (1/R) o(r \cdot u) + o(|r|^0), \quad (13 A)$$

$$s_{\pm}/r_{\pm} \cdot u_{\pm}^2 + \frac{1}{2} s_{\pm}^2 A_{\pm} = \pm 1/(2 R(1 - r \cdot \dot{u})) + (1/R) o(r \cdot u) + o(|r|), \quad (14 A)$$

$$s_{\pm}^2/r_{\pm} \cdot u_{\pm}^2 + \frac{1}{2} s_{\pm}^3 A_{\pm} = \frac{2}{3} + (1/R) o(r \cdot u) + o(|r|). \quad (15 A)$$

Putting these expressions into (6 A) we obtain finally the following series expansion for the quantities $\omega_{\pm}^{ij}$:

$$\begin{aligned} \omega_{\pm}^{ij} = & ((1 - r \cdot \dot{u})/(\mp R^3)) (1 - \frac{1}{2} R^2 \ddot{u}^2) r^{(i} u^{j)} \mp 1/(2 R(1 - r \cdot \dot{u})) (u^{(i} \ddot{u}^{j)} - r^{(i} \ddot{u}^{j)}) \\ & - \frac{2}{3} u^{(i} \ddot{u}^{j)} \mp (r \cdot u/R^3) (r^{(i} \dot{u}^{j)} - \frac{1}{2} r \cdot u u^{(i} \dot{u}^{j)}) + (1/R) o(r \cdot u) + o(|r|). \end{aligned} \quad (16 A)$$

Using this and the definition of φ^{ij} (24) we obtain immediately the desired expansion:

$$\begin{aligned} \varphi^{ij} = & ((1 - r \cdot \dot{u})/R^3) (1 - \frac{1}{2} R^2 \ddot{u}^2) r^{(i} u^{j)} + 1/(2 R(1 - r \cdot \dot{u})) (u^{(i} \ddot{u}^{j)} - r^{(i} \ddot{u}^{j)}) \\ & + (r \cdot u/R^3) (r^{(i} \dot{u}^{j)} - \frac{1}{2} r \cdot u u^{(i} \dot{u}^{j)}) + (1/R) o(r \cdot u) + o(|r|). \end{aligned} \quad (27)$$

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